Multi-Criteria Decision-Making Support with Belief Functions

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Classical decision-making methods with belief functions

Decision-making methods from a BBA (1)

Decision-making problem (DMP) FoD $\Theta = \{\theta_1, \ldots, \theta_n\}$ = set of possible solutions

Knowing a BBA $m(\cdot)$ over 2^Θ, how should I make my decision δ based on $m(\cdot)$?

In the classical DMP, we restrict $\delta \in \Theta$, i.e. the best decision $\hat{\theta}$ is a singleton of 2^Θ.

Classical DM methods

- Pessimistic Decision-Making attitude: Maximum of belief strategy

$$m(\cdot) \rightarrow Bel(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} Bel(\theta_i)$$

- Optimistic Decision-Making attitude: Maximum of plausibility strategy

$$m(\cdot) \rightarrow Pl(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} Pl(\theta_i)$$

- Compromise Decision-Making attitude: Maximum of probability strategy

$$m(\cdot) \rightarrow P(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} P(\theta_i)$$

where $P(\cdot) \in [Bel(\cdot), Pl(\cdot)]$ is a (subjective) proba measure approximated from the BBA $m(\cdot)$, typically obtained with a lossy transformation like BetP, or DSmP

Popular transformations of BBA to probability

Many methods exist, we only present the most popular – see [DSmT books] (Vol. 3)

Simplest method

Take the mass of each element of Θ and normalize, but it does not take into account partial ignorances

$$P_m(A) = \frac{m(A)}{\sum_{B \in \Theta} m(B)}$$

Method based on plausibility [Cobb Shenoy 2006]

Take the plausibility of each element of Θ and normalize, but it is inconsistent with belief interval

$$P_{Pl}(A) = \frac{Pl(A)}{\sum_{B \in \Theta} Pl(B)}$$

Pignistic probability [Smets 1990]

Redistribute the mass of partial ignorances **equally** to singletons included in them

$\Rightarrow$ higher entropy obtained with $BetP(\cdot)$

$$BetP(A) = \sum_{X \in 2^\Theta} \frac{|X \cap A|}{|A|} m(X)$$

DSmP probability [Dezert Smarandache 2008]

Redistribute mass of partial ignorances **proportionally to masses of singletons** included in them. $\epsilon > 0$ is a small parameter to prevent division by zero in some cases.

$\Rightarrow$ smaller entropy obtained with $DSmP(\cdot)$

$$DSmP_\epsilon(A) = \sum_{Y \in 2^\Theta} \frac{\sum_{\substack{Z \subseteq A \cap Y \\ |Z|=1}} m(Z) + \epsilon |A \cap Y|}{\sum_{\substack{Z \subseteq Y \\ |Z|=1}} m(Z) + \epsilon |Y|} m(Y)$$

Examples of probabilistic transformations

$P_{Pl}(.)$ is inconsistent with belief interval Consider $\Theta = \{A, B, C\}$, and the BBA

$$\begin{cases} m(A) = 0.2 \\ m(B \cup C) = 0.8 \end{cases} \Rightarrow \begin{cases} [Bel(A), Pl(A)] = [0.2, 0.2] \\ [Bel(B), Pl(B)] = [0, 0.8] \\ [Bel(C), Pl(C)] = [0, 0.8] \end{cases} \Rightarrow \begin{cases} P_{Pl}(A) = \frac{0.2}{0.2+0.8+0.8} \approx 0.112 < Bel(A) \\ P_{Pl}(B) = \frac{0.8}{0.2+0.8+0.8} \approx 0.444 \\ P_{Pl}(C) = \frac{0.8}{0.2+0.8+0.8} \approx 0.444 \end{cases}$$

Note: inconsistency also occurs with $P_{Bel}(.)$

Simple example for BetP and DSmP calculation

Consider $\Theta = \{A, B\}$, and $m(A) = 0.3$, $m(B) = 0.1$, $m(A \cup B) = 0.6$

$$\begin{cases} BetP(A) = m(A) + \frac{1}{2} m(A \cup B) = 0.3 + (0.6/2) = 0.6 \\ BetP(B) = m(B) + \frac{1}{2} m(A \cup B) = 0.1 + (0.6/2) = 0.4 \end{cases}$$

With DSmP the masses of singletons are used as a priori information to make the redistribution of the mass of ignorance (reinforcement principle)

$$\begin{cases} DSmP_{\epsilon=0.001}(A) = m(A) + \frac{m(A)+\epsilon}{m(A)+m(B)+2\epsilon} \cdot m(A \cup B) = 0.7492 \\ DSmP_{\epsilon=0.001}(B) = m(B) + \frac{m(B)+\epsilon}{m(A)+m(B)+2\epsilon} \cdot m(A \cup B) = 0.2508 \end{cases}$$

Shannon entropy (measure of randomness): $H(P) = -\sum_i p_i \log p_i$

$$H(DSmP) = 0.8125 \text{ bits} < H(BetP) = 0.9710 \text{ bits}$$

Thus, decision-making is made easier with DSmP because randomness is reduced

Decision-making methods from a BBA (2)

Decision-making based on distances [Han Dezert Yang 2014, Dezert et al. 2016]

A better theoretical approach for decision-making is to use a strict distance metric $d(\cdot,\cdot)$ between two BBAs and to make the decision by

$$\delta = \hat{X} = \arg\min_{X\in\mathcal{X}} d(m, m_X)$$

$\mathcal{X} = \{\text{admissible} X, X \in 2^\Theta\}$ is the set of possible admissible decisions we consider. For instance, if δ must be a singleton, then $\mathcal{X} = \Theta = \{\theta_1, \ldots, \theta_n\}$.

m_X is the BBA focused on X defined by $m_X(Y) = 0$ if $Y \neq X$, and $m_X(Y) = 1$ if $Y = X$

Few strict distance metrics are possible

- Jousselme distance: $d_J(m_1, m_2) \triangleq \sqrt{0.5 \cdot (m_1 - m_2)^T \mathbf{Jac} (m_1 - m_2)}$
- Euclidean d_{BI} distance: $d_{BI}^E(m_1, m_2) \triangleq \sqrt{\frac{1}{2^{|\Theta|-1}} \cdot \sum_{A\in 2^\Theta} d^I(BI_1(A), BI_2(A))^2}$
- Chebyshev d_{BI} distance: $d_{BI}^C(m_1, m_2) \triangleq \max_{A\in 2^\Theta} \left\{ d^I(BI_1(A), BI_2(A)) \right\}$

In practice, we recommend to use $d_{BI}^E(m_1, m_2)$ [Han Dezert Yang 2017]

Quality of the decision

$$q(\hat{X}) = 1 - \frac{d_{BI}(m, m_{\hat{X}})}{\sum_{X\in\mathcal{X}} d_{BI}(m, m_X)} \in [0, 1]$$

Higher is $q(\hat{X})$ more trustable is the decision $\delta = \hat{X}$

General mono-criteria decision-making problem

General mono-criteria decision-making problem

How to make a decision among several possible choices, based on some contexts ?

Problem modeling

$q \geqslant 2$ alternatives (choices) $\mathcal{A} = \{A_1, \ldots, A_q\}$
$n \geqslant 1$ states of nature (contexts) $\mathcal{S} = \{S_1, \ldots, S_n\}$

$$\mathbf{C} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & \ldots & S_j & \ldots & S_n \end{array} \\ \begin{bmatrix} C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\ \vdots & & \vdots & & \\ C_{i1} & \ldots & C_{ij} & \ldots & S_{in} \\ \vdots & & \vdots & & \\ C_{q1} & \ldots & C_{qj} & \ldots & C_{qn} \end{bmatrix} \end{array}$$

$\mathbf{C}$ is the benefit (payoff) matrix of the problem under consideration

Investment company example

An investment company wants to invest a given amount of money in the best option $A^* \in \mathcal{A} = \{A_1, A_2, A_3\}$, where $A_1 =$ car company, $A_2 =$ food company, and $A_3 =$ computer company. Several scenarios (states of nature) S_i are identified depending on national economical situation predictions, which provide the elements of the payoff matrix $\mathbf{C}$ according to a given criteria (growth analysis criterion by example).

Several decision-making frameworks are possible

- Decision under certainty
  If we know the **true** state of nature is S_j, take as decision $\delta = A^*$ with

$$A^* = A_{i*} \quad \text{with} \quad i^* = \arg\max_i \{C_{ij}\}$$

- Decision under risk
  If we know **all** probabilities $p_j = P(S_j)$ of the states of nature, compute the expected benefit $E[C_i] = \sum_j p_j C_{ij}$ of each A_i and take as decision $\delta = A^*$ with

$$A^* = A_{i*} \quad \text{with} \quad i^* = \arg\max_i \{E[C_i]\}$$

- Decision under ignorance
  If **we don't know the probabilities** $p_j = P(S_j)$ of the states of nature, use OWA (**O**rdered **W**eighted **A**veraging) approach [Yager 1988], or Cautious-OWA [Tacnet Dezert 2011], or Fuzzy-Cautious-OWA [Han Dezert Tacnet Han 2012]
- Decision under uncertainty
  If we have **only a BBA** over the states of the nature $\mathcal{S} = \{S_1, \ldots, S_n\}$ defined on the power set $2^{\mathcal{S}}$, we can use Yager extended OWA approach.

$$\mathbf{C} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{cccccc} S_1, p_1 & \dots & S_j, p_j & \dots & S_n, p_n \end{array} \\ \begin{bmatrix} C_{11} & \dots & C_{1j} & \dots & C_{1n} \\ \vdots & & \vdots & & \\ C_{i1} & \dots & C_{ij} & \dots & C_{in} \\ \vdots & & \vdots & & \\ C_{q1} & \dots & C_{qj} & \dots & C_{qn} \end{bmatrix} \end{array} \Rightarrow \mathrm{E}[\mathbf{C}] = \begin{bmatrix} \mathrm{E}[C_1] = \sum_j p_j C_{1j} \\ \vdots \\ \mathrm{E}[C_i] = \sum_j p_j C_{ij} \\ \vdots \\ \mathrm{E}[C_q] = \sum_j p_j C_{qj} \end{bmatrix}$$

Decision: A^* is the chosen alternative corresponding to highest expected benefit.

Example

$$\mathbf{C} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} S_1, p_1 = 1/4 & S_2, p_2 = 1/4 & S_3, p_3 = 1/2 \end{array} \\ \begin{bmatrix} 16 & 12 & 20 \\ 32 & 4 & 6 \\ 12 & 20 & 4 \\ 40 & 4 & 8 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} \mathrm{E}[C_1] = (1/4)16 + (1/4)12 + (1/2)20 = 17 \\ \mathrm{E}[C_2] = (1/4)32 + (1/4)4 + (1/2)6 = 12 \\ \mathrm{E}[C_3] = (1/4)12 + (1/4)20 + (1/2)4 = 10 \\ \mathrm{E}[C_4] = (1/4)40 + (1/4)4 + (1/2)8 = 15 \end{bmatrix}$$

Sorting the expected benefits by their decreasing values gives the ranking

$$A_1 \succ A_4 \succ A_2 \succ A_4$$

The decision to take is $A^* = A_1$

Decision under ignorance using OWA

The probabilities $p_j = P(S_j)$ of the states of the nature are unknown

$$C \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{cccccc} S_1, p_1 = ? & \dots & S_j, p_j = ? & \dots & S_n, p_n = ? \end{array} \\ \begin{bmatrix} C_{11} & \dots & C_{1j} & \dots & C_{1n} \\ \vdots & & \vdots & & \\ C_{i1} & \dots & C_{ij} & \dots & C_{in} \\ \vdots & & \vdots & & \\ C_{q1} & \dots & C_{qj} & \dots & C_{qn} \end{bmatrix} \end{array}$$

OWA (**O**rdered **W**eighted **A**veraging) method [Yager 1988]

1. Decisional attitude: choose the set of n weights $\mathbf{w} = [w_1 \ \dots \ w_n]$ with $\sum_j w_j = 1$
   - **Optimistic** (max benefit): $\mathbf{w} = [1\ 0 \dots 0]$
   - **Hurwicz** (a balance between min and max): $\mathbf{w} = [\alpha\ 0 \dots 0\ (1-\alpha)]$, typically $\alpha = 1/2$
   - **Normative** (equi weights): $\mathbf{w} = [\frac{1}{n} \dots \frac{1}{n}]$
   - **Pessimistic** (min benefit): $\mathbf{w} = [0 \dots 0\ 1]$
2. Evaluation: compute the weighted average of ordered benefits for each alternative

$$V_i = OWA(C_{i1}, \dots, C_{in}) = \sum_{j=1}^{n} w_j b_{ij}$$

   where b_{ij} is the j-th element/benefit among $\{C_{i1}, \dots, C_{in}\}$ and $\mathbf{b}_i = [b_{i1}\ b_{i2}\ \dots\ b_{in}]$ is the reordering of i-th row by decreasing values
3. Decision: take $\delta = A^*$ with

$$A^* = A_{i*} \quad \text{with} \quad i^* = \arg\max_i \{V_i\}$$

Example of decision under ignorance with OWA

The probabilities $p_j = P(S_j)$ of the states of the nature are unknown

$$\mathbf{C} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} S_1, p_1 =? & S_2, p_2 =? & S_3, p_3 =? & S_4, p_4 =? \end{array} \\ \begin{bmatrix} 10 & 0 & 20 & 30 \\ 1 & 10 & 20 & 30 \\ 30 & 10 & 2 & 5 \end{bmatrix} \end{array}$$

- OWA result with **optimistic** attitude $\mathbf{w} = [1\ 0\ 0\ 0] \rightarrow$ we take the max by row

$$\begin{cases} V_1 = OWA(10, 0, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 30 \\ V_2 = OWA(1, 10, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 30 \\ V_3 = OWA(30, 10, 2, 5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 30 \end{cases} \Rightarrow \text{No best choice exists}$$

- OWA result with **Hurwicz** attitude with $\alpha = 0.5 \Rightarrow \mathbf{w} = [(1/2)\ 0\ 0\ (1/2)]$

$$\begin{cases} V_1 = OWA(10, 0, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = (30/2) + (0/2) = 15 \\ V_2 = OWA(1, 10, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = (30/2) + (1/2) = 15.5 \\ V_3 = OWA(30, 10, 2, 5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = (30/2) + (2/2) = 16 \end{cases} \Rightarrow A_3 \text{ is the best choice}$$

- OWA result with **normative** attitude $\mathbf{w} = [(1/4)\ (1/4)\ (1/4)\ (1/4)]$

$$\begin{cases} V_1 = OWA(10, 0, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 60/4 = 15 \\ V_2 = OWA(1, 10, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 61/4 \\ V_3 = OWA(30, 10, 2, 5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 47/4 \end{cases} \Rightarrow A_2 \text{ is the best choice}$$

- OWA result with **pessimistic** attitude $\mathbf{w} = [0\ 0\ 0\ 1] \rightarrow$ we take the min by row

$$\begin{cases} V_1 = OWA(10, 0, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 0 \\ V_2 = OWA(1, 10, 20, 30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 1 \\ V_3 = OWA(30, 10, 2, 5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 2 \end{cases} \Rightarrow A_3 \text{ is the best choice}$$

Decision under uncertainty using OWA

Probas $p_j = P(S_j)$ of the states S_j **are unknown**, but we know a BBA $m(\cdot) : 2^S \mapsto [0, 1]$

$$C = [c_1 \ \ldots c_j \ \ldots c_n] \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{ccccc} S_1, p_1 =? & \ldots & S_j, p_j =? & \ldots & S_n, p_n =? \end{array} \\ \begin{bmatrix} C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \ldots & C_{ij} & \ldots & C_{in} \\ & & \vdots & & \\ C_{q1} & \ldots & C_{qj} & \ldots & C_{qn} \end{bmatrix} \end{array}$$

Method 1: Approximate $m(\cdot)$ by a proba measure $\Rightarrow$ decison-making under risk

Method 2: Extended OWA method [Yager 1988]

1. Decisional attitude: choose the decisional attitude (optimistic,pessimistic, etc)
2. Apply OWA on each sub-matrix $C(X_k)$ of benefits associated with the focal element X_k, $k = 1, \ldots, r$ of $m(\cdot)$ to get valuations $V_i(X_k)$, $i = 1, \ldots, q$

$$C(X_k) = [c_j | S_j \subseteq X_k]$$

3. Compute the generalized expected benefits for $i = 1, \ldots, q$

$$E[C_i] = \sum_{k=1}^{r} m(X_k) V_i(X_k)$$

4. Decision: take the decision $\delta = A^* = A_{i*}$ with $i^* = \arg\max_i \{E[C_i]\}$

Example of decision under uncertainty using OWA (1)

Probas $p_j = P(S_j)$ of the states S_j **are unknown**, but we know a BBA $m(\cdot) : 2^S \mapsto [0, 1]$

$$\mathbf{C} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1, p_1 =? & S_2, p_2 =? & S_3, p_3 =? & S_4, p_4 =? & S_5, p_5 =? \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array}$$

The uncertainty is modeled by a BBA with 3 focal elements as follows

BBA\FE	$X_1 \triangleq S_1 \cup S_3 \cup S_4$	$X_2 \triangleq S_2 \cup S_5$	$X_3 \triangleq S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$
$m(\cdot)$	0.6	0.3	0.1

Construction of benefit sub-matrices for each focal element of $m(\cdot)$

$$\mathbf{C}(X_1) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \end{array} \quad \mathbf{C}(X_2) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} S_2 & S_5 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \end{array} \quad \mathbf{C}(X_3) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array}$$

Example of decision under uncertainty using OWA (2)

Using **pessimistic** decisional attitude

- Apply OWA for each sub-matrix $\mathbf{C}(X_3)$, $k = 1, 2, 3$

$$\mathbf{C}(X_1) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_1) = OWA(7,12,13) = [0\,0\,1] \cdot [13\,12\,7]' = 7 \\ V_2(X_1) = OWA(12,5,11) = [0\,0\,1] \cdot [12\,11\,5]' = 5 \\ V_3(X_1) = OWA(9,3,10) = [0\,0\,1] \cdot [10\,9\,3]' = 3 \\ V_4(X_1) = OWA(6,11,15) = [0\,0\,1] \cdot [15\,11\,6]' = 6 \end{cases}$$

$$\mathbf{C}(X_2) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} S_2 & S_5 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_2) = OWA(5,6) = [0\,1] \cdot [6\,5]' = 5 \\ V_2(X_2) = OWA(10,2) = [0\,1] \cdot [10\,2]' = 2 \\ V_3(X_2) = OWA(13,9) = [0\,1] \cdot [13\,9]' = 9 \\ V_4(X_2) = OWA(9,4) = [0\,1] \cdot [9\,4]' = 4 \end{cases}$$

$$\mathbf{C}(X_3) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_3) = OWA(7,5,12,13,6) = [0\,0\,0\,0\,1] \cdot [13\,12\,7\,6\,5]' = 5 \\ V_2(X_3) = OWA(12,10,5,11,2) = [0\,0\,0\,0\,1] \cdot [12\,11\,10\,5\,2]' = 2 \\ V_3(X_3) = OWA(9,13,3,10,9) = [0\,0\,0\,0\,1] \cdot [13\,10\,9\,9\,3]' = 3 \\ V_4(X_3) = OWA(6,9,11,15,4) = [0\,0\,0\,0\,1] \cdot [15\,11\,9\,6\,4]' = 4 \end{cases}$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k) V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$E[C_1] = 0.6 \cdot 7 + 0.3 \cdot 5 + 0.1 \cdot 5 = 6.2$$
$$E[C_2] = 0.6 \cdot 5 + 0.3 \cdot 2 + 0.1 \cdot 2 = 3.8$$
$$E[C_3] = 0.6 \cdot 3 + 0.3 \cdot 9 + 0.1 \cdot 3 = 4.8$$
$$E[C_4] = 0.6 \cdot 6 + 0.3 \cdot 4 + 0.1 \cdot 4 = 5.2$$

- Take final decision with alternative having highest expected benefit $\rightarrow$ $A^* = A_1$

Example of decision under uncertainty using OWA (3)

Using **optimistic** decisional attitude

- Apply OWA for each sub-matrix $\mathbf{C}(X_3)$, $k = 1, 2, 3$

$$\mathbf{C}(X_1) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_1) = OWA(7, 12, 13) = [1\,0\,0] \cdot [13\,12\,7]' = 13 \\ V_2(X_1) = OWA(12, 5, 11) = [1\,0\,0] \cdot [12\,11\,5]' = 12 \\ V_3(X_1) = OWA(9, 3, 10) = [1\,0\,0] \cdot [10\,9\,3]' = 10 \\ V_4(X_1) = OWA(6, 11, 15) = [1\,0\,0] \cdot [15\,11\,6]' = 15 \end{cases}$$

$$\mathbf{C}(X_2) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} S_2 & S_5 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_2) = OWA(5, 6) = [1\,0] \cdot [6\,5]' = 6 \\ V_2(X_2) = OWA(10, 2) = [1\,0] \cdot [10\,2]' = 10 \\ V_3(X_2) = OWA(13, 9) = [1\,0] \cdot [13\,9]' = 13 \\ V_4(X_2) = OWA(9, 4) = [1\,0] \cdot [9\,4]' = 9 \end{cases}$$

$$\mathbf{C}(X_3) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{cases} V_1(X_3) = OWA(7, 5, 12, 13, 6) = [1\,0\,0\,0\,0] \cdot [13\,12\,7\,6\,5]' = 13 \\ V_2(X_3) = OWA(12, 10, 5, 11, 2) = [1\,0\,0\,0\,0] \cdot [12\,11\,10\,5\,2]' = 12 \\ V_3(X_3) = OWA(9, 13, 3, 10, 9) = [1\,0\,0\,0\,0] \cdot [13\,10\,9\,9\,3]' = 13 \\ V_4(X_3) = OWA(6, 9, 11, 15, 4) = [1\,0\,0\,0\,0] \cdot [15\,11\,9\,6\,4]' = 15 \end{cases}$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k) V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$E[C_1] = 0.6 \cdot 13 + 0.3 \cdot 6 + 0.1 \cdot 13 = 10.9$$
$$E[C_2] = 0.6 \cdot 12 + 0.3 \cdot 10 + 0.1 \cdot 12 = 11.4$$
$$E[C_3] = 0.6 \cdot 10 + 0.3 \cdot 13 + 0.1 \cdot 13 = 11.2$$
$$E[C_4] = 0.6 \cdot 15 + 0.3 \cdot 9 + 0.1 \cdot 15 = 13.2$$

- Take final decision with alternative having highest expected benefit $\rightarrow$ $A^* = A_4$

Advantage, limitation and improvement of OWA

Advantage of OWA

Very simple to apply

Limitation of OWA

The result strongly depends on the decisional attitude chosen when applying OWA
How to avoid this? → complicate methods exist to select weights (using entropy)

Improvements of OWA

Use jointly the two most extreme decisional attitudes (pessimistic and optimistic) to be more cautious, which can be done as follows

1. Applying OWA using Hurwicz attitude by taking $\alpha = 1/2$
   → a balance only between min and max benefit values
2. Applying modified OWA based on belief functions
   → we use all benefit values between min and max

   - Cautious OWA (COWA) [Tacnet Dezert 2011]

     Pessimistic and optimistic generalized expected benefits allow to build belief intervals, and to get BBAs that are combined with PCR6 to get combined BBA from which the final decision is taken.

   - Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]

     A version of COWA more efficient and more simple to implement

Cautious OWA for decision under ignorance or uncertainty

At first, apply OWA with pessimistic and optimistic attitudes to get bounds $[E^{min}[C_i], E^{max}[C_i]]$ of expected benefits of each alternative A_i

Main steps of Cautious OWA (COWA) [Tacnet Dezert 2011]

1. Normalization of exp. benefits intervals ($\div$ by max value) to get intervals in $[0, 1]$
2. Conversion of each interval in a BBA $m_i(A_i)$, $m_i(\bar{A}_i)$, $m_i(A_i \cup \bar{A}_i)$
3. Fusion of the q BBAs $m_i(\cdot)$, $i = 1, \ldots, q$ (by PCR6) to get the combined BBA $m(\cdot)$
4. Final decision drawn from $m(\cdot)$ by a chosen decision rule, for example by max BetP, max DSmP, or by min d_{BI})

Drawbacks of COWA

- High computational complexity of the combination (highly dependent on the number q of alternatives)
- In COWA, each expected interval is used as a SoE. However these intervals are jointly obtained which introduces a correlation between the sources, and which is harmful for the combination of BBAs.

Overcoming the drawbacks of COWA

→ Use Fuzzy-COWA approach, which is more efficient and simpler

Example of decision making using COWA (1)

Let consider the previous example with

BBA\FE	$X_1 \triangleq S_1 \cup S_3 \cup S_4$	$X_2 \triangleq S_2 \cup S_5$	$X_3 \triangleq S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$
$m(\cdot)$	0.6	0.3	0.1

and the benefit matrix

$$C = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{array}{l} [E^{min}[C_1] = 6.2, E^{max}[C_1] = 10.9] \\ [E^{min}[C_2] = 3.8, E^{max}[C_2] = 11.4] \\ [E^{min}[C_3] = 4.8, E^{max}[C_3] = 11.2] \\ [E^{min}[C_4] = 5.2, E^{max}[C_4] = 13.2] \end{array} \Rightarrow \begin{array}{l} [6.2/13.2, 10.9/13.2] \approx [0.47, 0.82] \\ [3.8/13.2, 11.4/13.2] \approx [0.29, 0.86] \\ [4.8/13.2, 11.2/13.2] \approx [0.36, 0.85] \\ [5.2/13.2, 13.2/13.2] \approx [0.39, 1.00] \end{array}$$

BBA construction from interval $[a, b] \subseteq [0, 1]$

$$\begin{cases} m_i(A_i) = a, \\ m_i(\bar{A}_i) = 1 - b \\ m_i(A_i \cup \bar{A}_i) = m_i(\Theta) = b - a \end{cases}$$

Alternatives A_i	$m_i(A_i)$	$m_i(\bar{A}_i)$	$m_i(A_i \cup \bar{A}_i)$
A_1	0.47	0.18	0.35
A_2	0.29	0.14	0.57
A_3	0.36	0.15	0.49
A_4	0.39	0	0.61

Example of decision making using COWA (2)

Fusion of BBAs (here with PCR5)

Focal Element	$m_{PCR5}(.)$
A_1	0.2488
A_2	0.1142
A_3	0.1600
A_4	0.1865
$A_1 \cup A_4$	0.0045
$A_2 \cup A_4$	0.0094
$A_1 \cup A_2 \cup A_4$	0.0236
$A_3 \cup A_4$	0.0075
$A_1 \cup A_3 \cup A_4$	0.0198
$A_2 \cup A_3 \cup A_4$	0.0374
$A_1 \cup A_2 \cup A_3 \cup A_4$	0.1883

Final decision (by max of Bel, BetP, DSmP or Pl)

A_i	$Bel(A_i)$	$BetP(A_i)$	$DSmP(A_i)$	$Pl(A_i)$
A_1	0.2488	0.3126	0.3364	0.4850
A_2	0.1142	0.1863	0.1623	0.3729
A_3	0.1600	0.2299	0.2242	0.4130
A_4	0.1865	0.2712	0.2771	0.4521

Final decision is $A^* = A_1$

Fuzzy Cautious OWA for decision under ignorance or uncertainty

At first, apply OWA with pessimistic and optimistic attitudes to get bounds $[\mathrm{E}^{\min}[\mathrm{C_i}], \mathrm{E}^{\max}[\mathrm{C_i}]]$ of expected benefits of each alternative A_i

Main steps of Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]

1. Normalize each column $\mathrm{E}^{\min}[\mathrm{C}]$ and $\mathrm{E}^{\max}[\mathrm{C}]$ **separately** to obtain $\mathrm{E}^{\mathrm{Fuzzy}}(\mathrm{C})$
2. Conversion of the two normalized columns, i.e. two Fuzzy Membership Functions (FMF), into **two pessimistic and optimistic BBAs** $\mathrm{m}_{\mathrm{Pess}}(\cdot)$ and $\mathrm{m}_{\mathrm{Opti}}(\cdot)$
3. Fusion of $\mathrm{m}_{\mathrm{Pess}}(\cdot)$ and $\mathrm{m}_{\mathrm{Opti}}(\cdot)$ to get a combined BBA $\mathrm{m}(\cdot)$
4. Final decision drawn from $\mathrm{m}(\cdot)$ by a chosen decision rule, for example by max of BetP, DSmP, or by min of d_{BI}

Advantages of FCOWA

- only 2 BBAs are involved in the combination $\Rightarrow$ only one fusion step is needed
- the BBAs in FCOWA (built by using alpha-cuts) are consonant support (FE are nested), which brings less computational complexity than with COWA
- good performances of FCOWA w.r.t. COWA
- good robustness of FCOWA to scoring errors w.r.t. COWA

Physical meaning

In FCOWA, the 2 SoE are pessimistic OWA and optimistic OWA. The combination result can be regarded as a tradeoff between these two attitudes.

Differences between COWA and FCOWA

The differences between COWA and FCOWA (on previous example):

Difference in normalization

COWA-ER

$$E^{Imp}[C] = \begin{bmatrix} [6.2/13.2; 10.9/13.2] \\ [3.8/13.2; 11.4/13.2] \\ [4.8/13.2; 11.2/13.2] \\ [5.2/13.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [0.47; 0.82] \\ [0.29; 0.86] \\ [0.36; 0.85] \\ [0.39; 1.00] \end{bmatrix}$$

FCOWA-ER

FMF1 FMF2

$$E^{Fuzzy}[C] = \begin{bmatrix} [6.2/6.2; 10.9/13.2] \\ [3.8/6.2; 11.4/13.2] \\ [4.8/6.2; 11.2/13.2] \\ [5.2/6.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [1.0000; 0.8258] \\ [0.6129; 0.8636] \\ [0.7742; 0.8485] \\ [0.8387; 1.0000] \end{bmatrix}$$

Difference in BBA modeling

COWA-ER

$$\begin{cases} m_i(A_i) = a, \\ m_i(\bar{A}_i) = 1 - b \\ m_i(A_i \cup \bar{A}_i) = m_i(\Theta) = b - a \end{cases}$$

FCOWA-ER

$$\begin{cases} B_j = \{A_i \in \Theta | \mu(A_i) \geq \alpha_j\} \\ m(B_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M} \end{cases}$$

alpha-cut

Principle of α-cut for BBA modeling in FCOWA

COWA

$$E^{Imp}[C] = \begin{bmatrix} [6.2/13.2; 10.9/13.2] \\ [3.8/13.2; 11.4/13.2] \\ [4.8/13.2; 11.2/13.2] \\ [5.2/13.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [0.47; 0.82] \\ [0.29; 0.86] \\ [0.36; 0.85] \\ [0.39; 1.00] \end{bmatrix}$$

Step 1

$$\begin{cases} m_i(A_i) = a, \\ m_i(\bar{A}_i) = 1 - b \\ m_i(A_i \cup \bar{A}_i) = m_i(\Theta) = b - a \end{cases}$$

Step 2

Combination + decision

Steps 3&4

FCOWA

$$E^{Fuzzy}[C] = \begin{bmatrix} [6.2/6.2; 10.9/13.2] \\ [3.8/6.2; 11.4/13.2] \\ [4.8/6.2; 11.2/13.2] \\ [5.2/6.2; 13.2/13.2] \end{bmatrix} \approx \begin{matrix} \mu_1(.) \quad \mu_2(.) \\ \begin{bmatrix} [1.0000; 0.8258] \\ [0.6129; 0.8636] \\ [0.7742; 0.8485] \\ [0.8387; 1.0000] \end{bmatrix} \end{matrix}$$

$$\begin{cases} B_j = \{A_i \in \Theta | \mu(A_i) \geq \alpha_j\} \\ m(B_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M} \end{cases}$$

alpha-cut

Focal Element	$m_{Pess}(.)$	Focal Element	$m_{Opti}(.)$
$A_1 \cup A_2 \cup A_3 \cup A_4$	0.6129	$A_1 \cup A_2 \cup A_3 \cup A_4$	0.8257
$A_1 \cup A_3 \cup A_4$	0.1613	$A_2 \cup A_3 \cup A_4$	0.0227
$A_1 \cup A_4$	0.0645	$A_2 \cup A_4$	0.0152
A_1	0.1613	A_4	0.1364

Combination + decision

Example: To compute $m_{Pess}(A_1 \cup A_4) = 0.0645$, we sort $\mu_1(.)$ in increasing order:

$$\mu_1(A_2) = 0.6129 = \alpha_1, \mu_1(A_3) = 0.7742 = \alpha_2, \mu_1(A_4) = 0.8387 = \alpha_3, \mu_1(A_1) = 1 = \alpha_4$$

Focal element $B_3 = \{A_i | \mu_1(A_i) \geq \alpha_3 = 0.8387\} = \{A_1 \cup A_4\}$, because only $\mu_1(A_1) \geq \alpha_3$ and $\mu_1(A_4) \geq \alpha_3$. Therefore

$$m_{Pess}(B_3) = m_{Pess}(A_1 \cup A_4) = \frac{\alpha_3 - \alpha_2}{\alpha_M} = \frac{0.8387 - 0.7742}{1} = 0.0645$$

On robustness of FCOWA on error scoring

Example: decision under ignorance with COWA and FCOWA

$$C = \begin{array}{c} \begin{matrix} S_1 & S_2 & S_3 & S_4 & S_5 \end{matrix} \\ \begin{bmatrix} 12 & 11 & 10 & 120 & 7 \\ 9 & 10 & 6 & 110 & 3 \\ 7 & 13 & 5 & 100 & 6 \\ 6 & 2 & 3 & 150 & 4 \end{bmatrix} \end{array} \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \qquad E[C] = \begin{bmatrix} [7, & 120] \\ [3, & 110] \\ [5, & 100] \\ [2, & 150] \end{bmatrix}$$

Rank-level Fusion

	A_1	A_2	A_3	A_4
S_1	1	2	3	4
S_2	2	3	1	4
S_3	1	2	3	4
S_4	2	3	4	1
S_5	1	4	2	3

COWA

$$E^{Imp}[C] = \begin{bmatrix} [0.0467, & 0.8000] \\ [0.0200, & 0.7333] \\ [0.0333, & 0.6667] \\ [0.0133, & 1.0000] \end{bmatrix}$$

Alternatives A_i	$m_i(A_i)$	$m_i(\bar{A}_i)$	$m_i(A_i \cup \bar{A}_i)$
A_1	0.0467	0.2000	0.7533
A_2	0.0200	0.2667	0.7133
A_3	0.0333	0.3333	0.6334
A_4	0.0133	0	0.9867

Focal Element	$BetP(.)$
A_1	0.2625
A_2	0.2152
A_3	0.2038
A_4	0.3185

FCOWA

$$E^{Fuzzy}[C] = \begin{bmatrix} [1.0000, & 0.8000] \\ [0.4286, & 0.7333] \\ [0.7143, & 0.6667] \\ [0.2857, & 1.0000] \end{bmatrix}$$

Focal Element	$m_{Pess}(.)$	Focal Element	$m_{Opti}(.)$
$A_1 \cup A_2 \cup A_3 \cup A_4$	0.2857	$A_1 \cup A_2 \cup A_3 \cup A_4$	0.6667
$A_1 \cup A_2 \cup A_3$	0.1429	$A_1 \cup A_3 \cup A_4$	0.0667
$A_1 \cup A_3$	0.2857	$A_3 \cup A_4$	0.0667
A_1	0.2857	A_1	0.1999

Focal Element	$BetP(.)$
A_1	0.5500
A_2	0.1056
A_3	0.2037
A_4	0.1407

The FCOWA method provides a decision $A^* = A_1$ which is consistent with what we obtain by rank-level fusion, contrariwise to what gives COWA

- no general proof of this good behavior of FCOWA has been proved so far
- impact of the normalization method on FCOWA performances not available yet

Methods for Multi-Criteria Decision-Making support

Classical Multi-Criteria Decision-Making (MCDM) problem

How to make a choice among several alternatives based on different criteria?

Problem modeling 1 $\Rightarrow$ using pairwise comparison matrices $\rightarrow$ **AHP methods**
We consider a set of criteria $C_1, \ldots, C_N$ with preferences of importance established from a pairwise comparison matrix (PCM) $\mathbf{M}$. For each criteria C_j, a set of preferences of the alternatives is established from a given **pairwise comparison matrix** $\mathbf{M}_j$.

Problem modeling 2 $\Rightarrow$ using directly the score matrix $\rightarrow$ **TOPSIS methods**

- A set of $M \geqslant 2$ alternatives $\mathcal{A} \triangleq \{A_1, \ldots, A_M\}$
- A set of $N > 1$ Criteria $\mathbf{C} \triangleq \{C_1, \ldots, C_N\}$
- A set of $N > 1$ criteria **importance weights** $W = \{w_1, \ldots, w_N\}$, with $w_j \in [0, 1]$ and $\sum_j w_j = 1$

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} \begin{array}{cccccc} C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N \end{array} \\ \begin{bmatrix} s_{11} & \ldots & s_{1j} & \ldots & s_{1N} \\ & & \vdots & & \\ s_{i1} & \ldots & s_{ij} & \ldots & s_{iN} \\ & & \vdots & & \\ s_{M1} & \ldots & s_{Mj} & \ldots & s_{MN} \end{bmatrix} \end{array}$$

$\mathbf{S}$ is the score matrix of the MCDM problem under consideration

Car example: How to buy a car based on some criteria (i.e. cost, safety, etc.)?

Methods for solving classical MCDM problem

Important remarks

- All methods developed so far suffer from rank reversal problem [Wang Luo 2009], which means that the rank is changed by adding (or deleting) an alternative in the problem. We consider rank reversal as very serious drawback.
- Most of existing methods require score normalization at first, except for ERV (Estimator Ranking Vector) method [Yin et al. 2013]. Normalization has been identified as one of the origins of rank reversal problem.
- There is no MCDM method which makes consensus among users, . . . but some are very popular
  - AHP (Analytic Hierarchy Process) method is very popular in operational research community but not exempt of problems
  - TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method is very popular **but the choice of normalization is disputed**

What we present

- AHP method and its extension DSm-AHP using belief functions [Saaty 1980, Dezert et al. 2010, Dezert Tacnet 2011]
- a new Belief-Function-based TOPSIS method called BF-TOPSIS to solve classical and non-classical MCDM problems [Dezert Han Yin 2016, Carladous et al. 2016]

Methods for Multi-Criteria Decision-Making support

- AHP and DSm-AHP methods

Classical AHP method for MCDM

AHP = **A**nalytic **H**ierarchy **P**rocess

AHP is a Multi-Criteria Decision-Making method developed by Thomas Saaty in 1980's based on the derivation of priorities from preferences.

Main steps of classical AHP method [Saaty 1980]

1. The multiple criteria $C_1, \ldots, C_N$ are ordered in a hierarchy of importance characterized by $\mathbf{w} = [w_1 \ \ldots \ w_N]$ such that $\sum_{j=1}^{N} w_j = 1$, obtained either through a given pairwise comparison matrix (PCM), or given directly.
2. For each criterion C_j, $j = 1, \ldots, N$, a set of preferences $\mathbf{w}(C_j)$ of the choice of alternatives is established from given pairwise comparison matrix $\mathbf{M}(C_j)$
3. Combine by the weighted arithmetic mean these preferences to get the global ranking of the alternatives
4. Final decision-making is based on the result of step 3 by selecting the most preferred alternative

Normalized Perron-Frobenius (NPF) eigen vectors (i.e. the eigen vector associated to largest eigen value) of Pairwise Comparison Matrices **are the keys of AHP** method.

Example for classical AHP method

Car selection example Θ = set of cars = $\{A, B, C, D\}$

We consider 3 criteria: C_1=Gasoil economy, C_2=Reliability, and C_3=style.

- Establishing importance of criteria from PCM (using NPF eigen vector)

$$\mathbf{M} = [M_{ij}] = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1/1 & 1/3 & 4/1 \\ 3/1 & 1/1 & 5/1 \\ 1/4 & 1/5 & 1/1 \end{bmatrix} \end{array} \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 0.2797 \\ 0.6267 \\ 0.0936 \end{bmatrix} \quad \Rightarrow \quad C_2 \succ C_1 \succ C_3$$

$M_{21} = 3/1$ means C_2 is 3 times as important as C_1

$M_{23} = 5/1$ means C_2 is 5 times as important as C_3

- Similarly, based on the given comparison matrices $\mathbf{M}(C_j)$ we get $\mathbf{w}(C_j)$
  For example, suppose we obtain from some PCM $\mathbf{M}(C_1)$, $\mathbf{M}(C_2)$ and $\mathbf{M}(C_3)$

$$\mathbf{W} \triangleq [\mathbf{w}(C_1)\ \mathbf{w}(C_2)\ \mathbf{w}(C_3)] = \begin{array}{c} \\ \text{Car } A \\ \text{Car B} \\ \text{Car C} \\ \text{Car D} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{w}(C_1) & \mathbf{w}(C_2) & \mathbf{w}(C_3) \end{array} \\ \begin{bmatrix} 0.2500 & 0.4733 & 0.1129 \\ 0.1304 & 0.0611 & 0.4435 \\ 0.5109 & 0.1832 & 0.0565 \\ 0.1087 & 0.2824 & 0.3871 \end{bmatrix} \end{array}$$

- Combination (by weighted arithmetic mean) to get final ranking vector $\mathbf{r}$

$$\mathbf{r} = \mathbf{W} \times \mathbf{w} = \begin{bmatrix} 0.2500 & 0.4733 & 0.1129 \\ 0.1304 & 0.0611 & 0.4435 \\ 0.5109 & 0.1832 & 0.0565 \\ 0.1087 & 0.2824 & 0.3871 \end{bmatrix} \times \begin{bmatrix} 0.2797 \\ 0.6267 \\ 0.0936 \end{bmatrix} = \begin{array}{c} \mathbf{r} \\ \text{Car } A \\ \text{Car B} \\ \text{Car C} \\ \text{Car D} \end{array} \begin{bmatrix} 0.3771 \\ 0.1163 \\ 0.2630 \\ 0.2436 \end{bmatrix} \Rightarrow A \succ C \succ D \succ B$$

- Final decision based on $\mathbf{r}$ vector: δ = Car A

Advantages and limitations of AHP for MCDM

Advantages

- quite easy to implement (toolboxes exist for eigen vector computation)
- easy to use
- pairwise comparison matrices are convenient for preference elicitation for experts

Limitations

- rank reversal problem
- does not take into account for uncertainties in the ranking process

Extension of AHP with DST [Beynon 2002]

- DS-AHP extends AHP using belief functions and Dempster-Shafer (DS) rule
- . . . but DS rule is questionable, and the importance discounting is not efficient

Extension of AHP with DSmT [Dezert et al. 2010, Dezert Tacnet 2011]

- DSm-AHP proposes a better rule of combination (PCR6)
- DSm-AHP proposes a more interesting importance discounting technique

DSm-AHP method for MCDM

DSm-AHP is an extension of Analytic Hierarchy Process (AHP) with using, PCR rules of combination, and the new importance discounting technique to take into account uncertainty in the ranking process

Main steps of DSm-AHP method [Dezert et al. 2010, Dezert Tacnet 2011]

1. Construction of uncertain comparison matrices. Take as BBA, the normalized Perron-Frobenius vector of each pairwise comparison matrix
2. Use PCR6 rule, to combine BBAs to get a final priority ranking vector $\mathbf{r}$
3. Make final decision by a chosen classical decision rule (i.e. max of Bel, max of Pl, max of BetP, max of DSmP, or min of $\mathrm{d_{BI}}$)

Advantages of DSm-AHP method

- better efficient rule of combination
- distinction between Shafer's reliability discounting and importance discounting

Drawbacks of DSm-AHP method

- rank reversal can occur with DS-AHP and DSm-AHP
- complicate to implement because of PCR6 general formula
- cannot work with many criteria and alternatives because of its too high complexity
- Solution → use BBA approximation techniques, and PCR6 rule sequentially

Reliability discounting versus importance discounting

Reliability discounting, $\alpha \in [0,1]$		**Importance discounting**, $\beta \in [0,1]$
$\begin{cases} m^{\alpha}(A) = \alpha \cdot m(A) \quad \forall A \neq \Theta \\ m^{\alpha}(\Theta) = \alpha \cdot m(\Theta) + (1-\alpha) \end{cases}$	$\neq$	$\begin{cases} m^{\beta}(A) = \beta \cdot m(A) \quad \forall A \neq \varnothing \\ m^{\beta}(\varnothing) = \beta \cdot m(\varnothing) + (1-\beta) \end{cases}$
$\alpha = 1 \Leftrightarrow$ SoE is 100% reliable		$\beta = 1 \Leftrightarrow$ SoE is 100% important
$\alpha = 0 \Leftrightarrow$ SoE is 100% unreliable		$\beta = 0 \Leftrightarrow$ SoE is not important

PCR5/6 fusion[1] **of importance discounted BBAs** (if $\beta_1 = \beta_2 = 0$, $m_{12}^{PCR5/6}(\Theta) \triangleq 1$)

$$m_{12}^{PCR5/6_{\varnothing}}(X) = m_{12}^{Conj,\beta_1\beta_2}(X) + \sum_{\substack{Y \in 2^{\Theta} \\ X \cap Y = \varnothing}} \left[\frac{m_1^{\beta_1}(X)^2 m_2^{\beta_2}(Y)}{m_1^{\beta_1}(X) + m_2^{\beta_2}(Y)} + \frac{m_2^{\beta_2}(X)^2 m_1^{\beta_1}(Y)}{m_2^{\beta_2}(X) + m_1^{\beta_1}(Y)} \right]$$

Because $m_{12}^{PCR5/6_{\varnothing}}(\varnothing) > 0$, a classical normalization applies, that is
$m_{12}^{PCR5/6}(\varnothing) = 0$, and $m_{12}^{PCR5/6}(X) = m_{12}^{PCR5/6_{\varnothing}}(X)/[1 - m_{12}^{PCR5/6_{\varnothing}}(\varnothing)]$ for $X \neq \varnothing$

Note: Dempster-Shafer rule **does not react** to importance discounting

[1] use general PCR6 formula for combining more than two BBAs, see [DSmT books], Vols. 2 & 3

Example for DSm-AHP method (1)

Car example $\{\text{Cars}\} = \Theta = \{A, B, C\}$, $\{\text{Criteria}\} = \{C_1 \triangleq \text{Economy}, C_2 \triangleq \text{Reliability}\}$

Suppose the two given pairwise comparisons matrices $\mathbf{M}(C_1)$ and $\mathbf{M}(C_2)$ are

$$\mathbf{M}(C_1) = \begin{array}{c} \\ A \\ B \cup C \\ \Theta \end{array} \begin{array}{c} \begin{array}{ccc} A & B \cup C & \Theta \end{array} \\ \begin{bmatrix} 1 & ? \rightarrow 0 & 1/3 \\ ? \rightarrow 0 & 1 & 2 \\ 3 & 1/2 & 1 \end{bmatrix} \end{array} \Rightarrow \mathbf{w}(C_1) \approx \begin{bmatrix} 0.0889 \\ 0.5337 \\ 0.3774 \end{bmatrix} = \begin{bmatrix} m_1(A) \\ m_1(B \cup C) \\ m_1(\Theta) \end{bmatrix}$$

$$\mathbf{M}(C_2) = \begin{array}{c} \\ A \\ B \\ A \cup C \\ B \cup C \end{array} \begin{array}{c} \begin{array}{cccc} A & B & A \cup C & B \cup C \end{array} \\ \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1/2 & 1 & 1/2 & 1/5 \\ 1/4 & 2 & 1 & ? \rightarrow 0 \\ 1/3 & 5 & ? \rightarrow 0 & 1 \end{bmatrix} \end{array} \Rightarrow \mathbf{w}(C_2) \approx \begin{bmatrix} 0.5002 \\ 0.1208 \\ 0.1222 \\ 0.2568 \end{bmatrix} = \begin{bmatrix} m_2(A) \\ m_2(B) \\ m_2(A \cup C) \\ m_2(B \cup C) \end{bmatrix}$$

Suppose the two criteria have **same full importances**, i.e. $\beta_1 = 1$ and $\beta_2 = 1$

We apply directly PCR6 rule

FE of 2^Θ	$m_1(\cdot)$	$m_2(\cdot)$	$m_{12}^{PCR5/6}(\cdot)$
$\varnothing$	0	0	0
A	0.0889	0.5002	0.3837
B	0	0	0.1162
$A \cup B$	0	0.1208	0
C	0	0	0.0652
$A \cup C$	0	0.1222	0.0461
$B \cup C$	0.5337	0.2568	0.3887
$A \cup B \cup C$	0.3774	0	0

We take the final decision according a chosen decision rule from $m_{12}^{PCR5/6}(\cdot)$

FE of 2^Θ	$Bel(\cdot)$	$BetP(\cdot)$	$Pl(\cdot)$
A	0.3837	0.4068	0.4298
B	0.1162	0.3105	0.5049
C	0.0652	0.2826	0.5000

Example for DSm-AHP method (2)

Car example again with **different importances** $\beta_1 = 0.25$ and $\beta_2 = 0.75$

With DSm-AHP

We apply importance discounting to derive $m_1^{\beta_1}(\cdot)$ and $m_2^{\beta_2}(\cdot)$, apply PCR5/6 rule to get $m_{12}^{PCR5/6_\varnothing}(\cdot)$ and normalize to get $m_{12}^{PCR5/6}(\cdot)$

FE of 2^Θ	$m_1(\cdot)$	$m_2(\cdot)$	$m_1^{\beta_1}(.)$	$m_2^{\beta_2}(\cdot)$	$m_{12}^{PCR5/6_\varnothing}(\cdot)$	$m_{12}^{PCR5/6}(\cdot)$
$\varnothing$	0	0	0.7500	0.2500	0.6558	0
A	0.0889	0.5002	0.0222	0.3751	0.1794	0.5213
B	0	0	0	0	0.0121	0.0351
$A \cup B$	0	0.1208	0	0.0906	0.0159	0.0461
C	0	0	0	0	0.0122	0.0355
$A \cup C$	0	0.1222	0	0.0917	0.0161	0.0469
$B \cup C$	0.5337	0.2568	0.1334	0.1926	0.1020	0.2963
$A \cup B \cup C$	0.3774	0	0.0944	0	0.0065	0.0188

With classic AHP (by simple componentwise weighted averaging)

$$m_{12}^{AHP}(\cdot) = [m_1(\cdot)\ m_2(\cdot)] \times \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0.0889 & 0.5002\\ 0 & 0\\ 0 & 0.1208\\ 0 & 0\\ 0 & 0.1222\\ 0.5337 & 0.2568\\ 0.3774 & 0\end{bmatrix} \times \begin{bmatrix}0.25\\ 0.75\end{bmatrix} \approx \begin{bmatrix}0\\ 0.3974\\ 0\\ 0.0906\\ 0\\ 0.0916\\ 0.3260\\ 0.0944\end{bmatrix}$$

Example for DSm-AHP method (3)

Car example again with **different importances** $\beta_1 = 0.25$ and $\beta_2 = 0.75$

$\Rightarrow$ DSm-AHP reduces the uncertainty of the result $U(X) = Pl(X) - Bel(X)$

Decision drawn from classical AHP using $m_{12}^{AHP}(\cdot) \rightarrow \delta = A$

FE of 2^Θ	$Bel(\cdot)$	$BetP(\cdot)$	$Pl(\cdot)$	$U(\cdot)$
A	0.3974	0.5200	0.6741	0.2767
B	0	0.2398	0.5110	0.5110
C	0	0.2403	0.5121	0.5121

Decision drawn from DSm-AHP using $m_{12}^{PCR5/6}(\cdot) \rightarrow \delta = A$

FE of 2^Θ	$Bel(\cdot)$	$BetP(\cdot)$	$Pl(\cdot)$	$U(\cdot)$
A	0.5213	0.5741	0.6331	0.1118
B	0.0351	0.2126	0.3963	0.3612
C	0.0355	0.2134	0.3974	0.3619

In this example AHP and DSm-AHP provide the same decision, but DSm-AHP offers a better precision (less uncertainty) on the result

Methods for Multi-Criteria Decision-Making support

- TOPSIS and BF-TOPSIS methods

TOPSIS = **T**echnique for **O**rder **P**reference by **S**imilarity to **I**deal **S**olution

Classical TOPSIS method [Hwang Yoon 1981]

1. Build the normalized score matrix $\mathbf{R} = [R_{ij}] = [S_{ij}/\sqrt{\sum_i S_{ij}^2}]$
2. Calculate the weighted normalized decision matrix $\mathbf{D} = [w_j \cdot R_{ij}]$
3. Determine the positive (best) ideal solution A^{best} by **taking the best/max value** in each column of $\mathbf{D}$
4. Determine the negative (worst) ideal solution A^{worst} by **taking the worst/min value** in each column of $\mathbf{D}$
5. Compute L2-distances $d(A_i, A^{best})$ of A_i, (i=1,...,M) to A^{best}, and $d(A_i, A^{worst})$ of A_i to A^{worst}
6. Calculate the relative closeness of A_i to best ideal solution A^{best} by

$$C(A_i, A^{best}) \triangleq \frac{d(A_i, A^{worst})}{d(A_i, A^{worst}) + d(A_i, A^{best})}$$

When $C(A_i, A^{best}) = 1$, its means that $A_i = A^{best}$ because $d(A_i, A^{best}) = 0$

When $C(A_i, A^{best}) = 0$, its means that $A_i = A^{worst}$ because $d(A_i, A^{worst}) = 0$

7. Rank alternatives A_i according to $C(A_i, A^{best})$ in descending order, and select the highest preferred solution

Example for classical TOPSIS method

A very simple example for TOPSIS

$$\mathbf{S} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} C_1, w_1 = 1/2 & C_2, w_2 = 1/2 \end{array} \\ \begin{bmatrix} 6 & 2 \\ 3 & 5 \\ 4 & 4 \end{bmatrix} \end{array}$$

1. Step 1 & 2 (normalization & columns weighting):

$$\mathbf{R} = [S_{ij}/\sqrt{\sum_i S_{ij}^2}] \Rightarrow \mathbf{R} = \begin{array}{c} \begin{array}{cc} C_1, 1/2 & C_2, 1/2 \end{array} \\ \begin{bmatrix} 0.7682 & 0.2981 \\ 0.3841 & 0.7454 \\ 0.5121 & 0.5963 \end{bmatrix} \end{array} \Rightarrow \mathbf{D} = \begin{bmatrix} 0.3841 & 0.1491 \\ 0.1921 & 0.3727 \\ 0.2561 & 0.2981 \end{bmatrix}$$

2. Step 3 & 4 (best and worst solutions) $A^{best} = [0.3841\ 0.3727]$, $A^{worst} = [0.1921\ 0.1491]$
3. Step 5 (L_2-distance of A_i to A^{best} and to A^{worst}):

$$\begin{array}{c} \\ A_1 = [0.3841\ 0.1491] \\ A_2 = [0.1921\ 0.3727] \\ A_3 = [0.2561\ 0.2981] \end{array} \begin{array}{c} \begin{array}{cc} A^{best} = [0.3841\ 0.3727] & A^{worst} = [0.1921\ 0.1491] \end{array} \\ \begin{bmatrix} d(A_1, A^{best}) = 0.2236 & d(A_1, A^{worst}) = 0.1921 \\ d(A_2, A^{best}) = 0.1921 & d(A_2, A^{worst}) = 0.2236 \\ d(A_3, A^{best}) = 0.1482 & d(A_3, A^{worst}) = 0.1622 \end{bmatrix} \end{array}$$

4. Step 6 (relative closeness of A_i to A^{best}): $C(A_i, A^{best}) \triangleq \frac{d(A_i, A^{worst})}{d(A_i, A^{worst}) + d(A_i, A^{best})}$

$$C(A_1, A^{best}) = 0.4620 \qquad C(A_2, A^{best}) = 0.5380 \qquad C(A_3, A^{best}) = 0.5227$$

5. Step 7 (ranking by decreasing order of $C(A_i, A^{best})$): $A_2 \succ A_3 \succ A_1$

Based on TOPSIS, the decision δ to make is $\delta = A_2$

BF-TOPSIS method for MCDM

BF-TOPSIS is a TOPSIS-alike method based on belief functions [Dezert Han Yin 2016]

Advantages of BF-TOPSIS

- no need for ad-hoc choice of scores normalization
- relatively simple to implement
- more robust to rank reversal phenomena (although not exempt)

Main problem to overcome

Working with belief functions requires the construction of BBAs. How to build efficiently BBAs from the score values

Solution → see next slides

Four BF-TOPSIS methods available with different complexity

1. BF-TOPSIS1: smallest complexity
2. BF-TOPSIS2: medium complexity
3. BF-TOPSIS3: high complexity (because of PCR6 fusion rule)
4. BF-TOPSIS4: high complexity (because of ZPCR6 fusion rule)

BF-TOPSIS for working with imprecise scores presented in [Dezert Han Tacnet 2017]

BBA construction for BF-TOPSIS (1)

- Positive support of A_i based on all scores values of a criteria C_j

$$Sup_j(A_i) \triangleq \sum_{k \in \{1,...M\} | S_{kj} \leqslant S_{ij}} |S_{ij} - S_{kj}|$$

$Sup_j(A_i)$ measures how much A_i is better (higher) than other alternatives

- Negative support of A_i based on all scores values of a criteria C_j

$$Inf_j(A_i) \triangleq - \sum_{k \in \{1,...M\} | S_{kj} \geqslant S_{ij}} |S_{ij} - S_{kj}|$$

$Inf_j(A_i)$ measures how much A_i is worse (lower) than other alternatives

Important inequality see proof in [Dezert Han Yin 2016]

$$\frac{Sup_j(A_i)}{A^j_{max}} \leqslant 1 - \frac{Inf_j(A_i)}{A^j_{min}}$$

iff $A^j_{max} \triangleq \max_i Sup_j(A_i)$ and $A^j_{min} \triangleq \min_i Inf_j(A_i)$ are different from zero.

BBA construction for BF-TOPSIS (2)

Reminder

$$\frac{Sup_j(A_i)}{A^j_{max}} \leqslant 1 - \frac{Inf_j(A_i)}{A^j_{min}}$$

Belief function modeling

$$Bel_{ij}(A_i) \triangleq \frac{Sup_j(A_i)}{A^j_{max}} \quad \text{and} \quad Bel_{ij}(\bar{A}_i) \triangleq \frac{Inf_j(A_i)}{A^j_{min}}$$

If $A^j_{max} = 0$, we set $Bel_{ij}(X_i) = 0$
If $A^j_{min} = 0$, we set $Pl_{ij}(A_i) = 1$ so that $Bel_{ij}(\bar{A}_i) = 0$

By construction, $0 \leqslant Bel_{ij}(A_i) \leqslant (Pl_{ij}(A_i) = 1 - Bel_{ij}(\bar{A}_i)) \leqslant 1$

BBA construction from Belief Interval

From $[Bel_{ij}(A_i), Pl_{ij}(A_i)]$, one gets the $M \times N$ BBAs matrix $\mathbf{M} = [m_{ij}(\cdot)]$ by taking

$$\begin{aligned} m_{ij}(A_i) &= Bel_{ij}(A_i) \\ m_{ij}(\bar{A}_i) &= Bel_{ij}(\bar{A}_i) = 1 - Pl_{ij}(A_i) \\ m_{ij}(A_i \cup \bar{A}_i) &= Pl_{ij}(A_i) - Bel_{ij}(A_i) \end{aligned}$$

Advantages of this BBA construction

1. if all S_{ij} are the same for a given column, we get $\forall A_i$, $Sup_j(A_i) = Inf_j(A_i) = 0$ and therefore $m_{ij}(A_i \cup \bar{A}_i) = 1$ which is the vacuous BBA, which makes sense.
2. it is invariant to the bias and scaling effects of score values. Indeed, if S_{ij} are replaced by $S'_{ij} = a \cdot S_{ij} + b$, with a scale factor $a > 0$ and a bias $b \in \mathbb{R}$, then $m_{ij}(\cdot)$ and $m'_{ij}(\cdot)$ remain equal.
3. if a numerical value S_{ij} is missing or indeterminate, then we use the vacuous belief assignment $m_{ij}(A_i \cup \bar{A}_i) = 1$.
4. We can also discount the BBA $m_{ij}(\cdot)$ by a reliability factor using the classical Shafer's discounting method if one wants to express some doubts on the reliability of $m_{ij}(\cdot)$.

In summary

From $[S_{ij}]$, we know how to build the matrix $\mathbf{M} = [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]$

How to use these BBAs to rank A_i to make a decision? → BF-TOPSIS methods

Steps of BF-TOPSIS1 [Dezert Han Yin 2016]

1. From $\mathbf{S}$, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$
2. Set $m_{ij}^{best}(A_i) \triangleq 1$, and $m_{ij}^{worst}(\bar{A}_i) \triangleq 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{best})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{worst})$ to ideal solutions.
3. Compute the weighted average distances of A_i to ideal solutions

$$d^{best}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{best})$$

$$d^{worst}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{worst})$$

4. Compute the relative closeness of A_i with respect to ideal best solution A^{best}

$$C(A_i, A^{best}) \triangleq \frac{d^{worst}(A_i)}{d^{worst}(A_i) + d^{best}(A_i)}$$

5. Rank A_i by $C(A_i, A^{best})$ in descending order.

Steps of BF-TOPSIS2 [Dezert Han Yin 2016]

1. From $\mathbf{S}$, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$
2. Set $m_{ij}^{best}(A_i) \triangleq 1$, and $m_{ij}^{worst}(\bar{A}_i) \triangleq 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{best})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{worst})$ to ideal solutions.
3. For each criteria C_j, compute the relative closeness of A_i to best ideal solution A^{best} by

$$C_j(A_i, A^{best}) \triangleq \frac{d_{BI}^{E}(m_{ij}, m_{ij}^{worst})}{d_{BI}^{E}(m_{ij}, m_{ij}^{worst}) + d_{BI}^{E}(m_{ij}, m_{ij}^{best})}$$

4. Compute the weighted average of $C_j(A_i, A^{best})$ by

$$C(A_i, A^{best}) \triangleq \sum_{j=1}^{N} w_j \cdot C_j(A_i, A^{best})$$

5. Rank A_i by $C(A_i, A^{best})$ in descending order.

Steps of BF-TOPSIS3 [Dezert Han Yin 2016]

1. Compute BBAs $m_{ij}(A_i)$, $m_{ij}(\bar{A}_i)$ and $m_{ij}(A_i \cup \bar{A}_i)$ and apply importance discounting of each BBA with weight w_j, see [Smarandache Dezert Tacnet 2010]
2. For each A_i, fuse the discounted BBAs with PCR6 to get BBAs $m_i^{PCR6}(\cdot)$
3. Set $m_i^{best}(A_i) \triangleq 1$, and $m_i^{worst}(\bar{A}_i) \triangleq 1$. Compute distances

$$d^{best}(A_i) \triangleq d_{BI}^{E}(m_i^{PCR6}, m_i^{best})$$

$$d^{worst}(A_i) \triangleq d_{BI}^{E}(m_i^{PCR6}, m_i^{worst})$$

4. Compute the relative closeness of A_i, $i = 1, \ldots, M$, with respect to ideal best solution A^{best}

$$C(A_i, A^{best}) \triangleq \frac{d^{worst}(A_i)}{d^{worst}(A_i) + d^{best}(A_i)}$$

5. Rank A_i by $C(A_i, A^{best})$ in descending order.

BF-TOPSIS4 method

Same as BF-TOPSIS3, but PCR6 rule is replaced by ZPCR6 rule (i.e. PCR6 rule including Zhang's degree of intersection) [Smarandache Dezert 2015]

BF-TOPSIS methods **are consistent** with direct ranking in mono-criteria case

Example (Mono-criteria) Preference order $\rightarrow$ greater value is better

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C_1 \\ \left[\begin{array}{c} 10 \\ 20 \\ -5 \\ 0 \\ 100 \\ -11 \\ 0 \end{array}\right. \end{array} \Rightarrow \mathbf{M} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} m_{i1}(A_i) \quad m_{i1}(\bar{A}_i) \quad m_{i1}(A_i \cup \bar{A}_i) \\ \left[\begin{array}{ccc} 0.0955 & 0.5236 & 0.3809 \\ 0.1809 & 0.4188 & 0.4003 \\ 0.0102 & 0.8115 & 0.1783 \\ 0.0273 & 0.6806 & 0.2921 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0.0273 & 0.6806 & 0.2921 \end{array}\right. \end{array} \Rightarrow \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C(A_i, A^{\text{best}}) \\ \left[\begin{array}{c} 0.1130 \\ 0.1948 \\ 0.0257 \\ 0.0485 \\ 1.0000 \\ 0 \\ 0.0485 \end{array}\right] \end{array}$$

Results

Ranking methods	**Preferences order**
By direct ranking	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By BF-TOPSIS	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By DS fusion	$A_5 \succ (A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_6 \sim A_7)$
By PCR6 fusion	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ (A_3 \sim A_6 \sim A_7)$

Rankings resulting of DS and PCR6 fusion of the BBAs do not match with direct ranking even in mono criteria case because of strong dependencies between BBAs in their construction.

On consistency of BF-TOPSIS methods (2)

In this example, we have $Score(A_5) >> Score(A_2)$

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C_1 \\ \left[\begin{array}{c} 10 \\ 20 \\ -5 \\ 0 \\ 100 \\ -11 \\ 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C(A_i, A^{best}) \\ \left[\begin{array}{c} 0.1130 \\ 0.1948 \\ 0.0257 \\ 0.0485 \\ 1.0000 \\ 0 \\ 0.0485 \end{array}\right] \end{array} \Rightarrow A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$$

Let's modify the example with $Score(A_5) \sim Score(A_2)$

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C_1 \\ \left[\begin{array}{c} 10 \\ 20 \\ -5 \\ 0 \\ 21 \\ -11 \\ 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} C(A_i, A^{best}) \\ \left[\begin{array}{c} 0.5072 \\ 0.9472 \\ 0.0675 \\ 0.1584 \\ 1.0000 \\ 0 \\ 0.1584 \end{array}\right] \end{array} \Rightarrow A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$$

We see that A_2 is very close to ideal best solution, even if final result is unchanged.

BF-TOPSIS when all scores are the same

When all scores are the same

$\Rightarrow$ all BBAs are the same and equal to the vacuous BBA

$\Rightarrow$ all closeness measures to best ideal solution are equal

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} C_1 \\ \left[\begin{array}{c} s \\ \vdots \\ s \\ \vdots \\ s \end{array}\right] \end{array} \Rightarrow \mathbf{M} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} m_{i1}(A_i \cup \bar{A}_i) \\ \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} C(A_i, A^{\text{best}}) \\ \left[\begin{array}{c} c \\ \vdots \\ c \\ \vdots \\ c \end{array}\right] \end{array}$$

Conclusion: No specific choice can be drawn, which is perfectly normal.

Multi-Criteria example [Wang Luo 2009]

We consider 5 alternatives, and 4 criteria

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1, \frac{1}{6} & C_2, \frac{1}{3} & C_3, \frac{1}{3} & C_4, \frac{1}{6} \end{array} \\ \begin{bmatrix} 36 & 42 & 43 & 70 \\ 25 & 50 & 45 & 80 \\ 28 & 45 & 50 & 75 \\ 24 & 40 & 47 & 100 \\ 30 & 30 & 45 & 80 \end{bmatrix} \end{array}$$

Rank reversal with TOPSIS

Set of alternatives	TOPSIS
$\{A_1, A_2, A_3\}$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal

Rank reversal with BF-TOPSIS

Set of alternatives	BF-TOPSIS1 & BF-TOPSIS2	BF-TOPSIS3 & BF-TOPSIS4
$\{A_1, A_2, A_3\}$	$A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_3 \succ A_2 \succ A_4 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal	No rank reversal

Car selection example

How to buy a car among 4 possible choices, and based on 5 different criteria with weights $w_1 = 5/17$, $w_2 = 4/17$, $w_3 = 4/17$, $w_4 = 1/17$, and $w_5 = 3/17$

- C_1 = price (in €); the least is the best
- C_2 = fuel consumption (in L/km); the least is the best
- C_3 = CO_2 emission (in g/km); the least is the best
- C_4 = fuel tank volume (in L); the biggest is the best
- C_5 = trunk volume (in L); the biggest is the best

Building the score matrix from `http://www.choisir-sa-voiture.com`

$$\mathbf{S} \triangleq \begin{array}{l} \\ A_1 = \text{TOYOTA YARIS 69 VVT-i Tendance} \\ A_2 = \text{SUZUKI SWIFT MY15 1.2 VVT So'City} \\ A_3 = \text{VOLKSWAGEN POLO 1.0 60 Confortline} \\ A_4 = \text{OPEL CORSA 1.4 Turbo 100 ch Start/Stop Edition} \end{array} \begin{array}{c} \begin{array}{ccccc} C_1, \frac{5}{17} & C_2, \frac{4}{17} & C_3, \frac{4}{17} & C_4, \frac{1}{17} & C_5, \frac{3}{17} \end{array} \\ \begin{bmatrix} 15000 & 4.3 & 99 & 42 & 73 \\ 15290 & 5.0 & 116 & 42 & 892 \\ 15350 & 5.0 & 114 & 45 & 952 \\ 15490 & 5.3 & 123 & 45 & 1120 \end{bmatrix} \end{array}$$

A_1 **is the expected best choice** because the 3 most important criteria meet their best values for car A_1.

With classical TOPSIS $A_4 \succ A_1 \succ A_3 \succ A_2$ (counter-intuitive)

With all BF-TOPSIS methods $A_1 \succ A_3 \succ A_2 \succ A_4$ (which fits with what we expect)

Best student prize example

How to give the best student prize awards among 4 students $\{A_1, A_2, A_3, A_4\}$, and based on 10 different criteria with equal importance ($w_j = 1/10$, $j = 1, \ldots, 10$) ?

	A_1	A_2	A_3	A_4
$C_1 \triangleq$ Math	90	80	70	60
$C_2 \triangleq$ Arts	90	80	70	60
$C_3 \triangleq$ English	90	80	70	60
$C_4 \triangleq$ Geography	90	80	70	60
$C_5 \triangleq$ Physics	90	80	70	75
$C_6 \triangleq$ Music	90	80	70	95
$C_7 \triangleq$ History	80	90	70	85
$C_8 \triangleq$ Chemistry	80	90	70	85
$C_9 \triangleq$ Biology	80	90	70	85
$C_{10} \triangleq$ Long jump	3.5m	3.7m	4.0m	3.6m

BF-TOPSIS results

	Considering 3 students $\{A_1, A_2, A_3\}$ only		**Considering the 4 students**	
Methods	Ranking vectors	Preferences orders	Ranking vectors	Preferences orders
ERV	$[0.748, 0.636, 0.188]$	$A_1 \succ A_2 \succ A_3$	$[0.620, 0.636, 0.248, 0.386]$	$A_2 \succ A_1 \succ A_4 \succ A_3$
BF-TOPSIS1	$[0.729, 0.594, 0.100]$	$A_1 \succ A_2 \succ A_3$	$[0.675, 0.620, 0.195, 0.320]$	$A_1 \succ A_2 \succ A_4 \succ A_3$
BF-TOPSIS2	$[0.731, 0.597, 0.100]$	$A_1 \succ A_2 \succ A_3$	$[0.677, 0.622, 0.194, 0.319]$	$A_1 \succ A_2 \succ A_4 \succ A_3$
BF-TOPSIS3	$[0.803, 0.736, 0.100]$	$A_1 \succ A_2 \succ A_3$	$[0.766, 0.775, 0.158, 0.288]$	$A_2 \succ A_1 \succ A_4 \succ A_3$
BF-TOPSIS4	$[0.803, 0.736, 0.100]$	$A_1 \succ A_2 \succ A_3$	$[0.766, 0.775, 0.158, 0.288]$	$A_2 \succ A_1 \succ A_4 \succ A_3$

ERV, BFTOPSIS3, and BFTOPSIS4 exhibit rank reversal

BFTOPSIS1and BFTOPSIS2 work fine here (no rank reversal)

Extension of BF-TOPSIS to work with imprecise scores

J. Dezert, D. Han, J.-M. Tacnet, Multi-Criteria Decision-Making Support with Imprecise Scores and BF-TOPSIS, in Proc. of Fusion 2017, Xi'an, July 2017.

Example: An investor wants to invest some money in a company to make the highest profit:

- Four alternatives (companies) $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$
- Two criteria are considered:
  - C_1=risk analysis (the min is better) with weight $w_1 = 0.6$
  - C_2=growth analysis (the max is better) with $w_2 = 0.4$

$$\mathbf{S} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} C_1(\text{in \%}), w_1 = 0.6 & C_2(\text{in \%}), w_2 = 0.4 \end{array} \\ \begin{bmatrix} [8,12] & [8,10] \\ [17,19] & [15,19] \\ [3,5] & [8,12] \\ [4,8] & [5,7] \end{bmatrix} \end{array}$$

Imp-BF-TOPSIS results

Method	$[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$	Preferences
Imp-BF-TOPSIS1	$[1.4939, 1.0013, 1.9396, 1.5652]$	$A_3 \succ A_4 \succ A_1 \succ A_2$
Imp-BF-TOPSIS2	$[1.4968, 0.9323, 2.0535, 1.5175]$	$A_3 \succ A_4 \succ A_1 \succ A_2$
Imp-BF-TOPSIS3	$[1.5300, 1.3487, 1.5639, 1.5574]$	$A_3 \succ A_4 \succ A_1 \succ A_2$
Imp-BF-TOPSIS4	$[1.5093, 1.4686, 1.5101, 1.5120]$	$A_4 \succ A_3 \succ A_1 \succ A_2$

Imp-BF-TOPSIS3 and Imp-BF-TOPSIS4 have difficulties to provide very distinct likelihoods because of IA calculus pitfall in implementation of PCR6 and ZPCR6 rules.

Non classical MCDM problem

Non-Classical Multi-Criteria Decision-Making problem

How to make a choice in $\mathcal{A}$ from multi-criteria scores expressed on power-set of $\mathcal{A}$?

$$
\mathbf{S} \triangleq
\begin{array}{l}
X_i \in 2^{\mathcal{A}} \\
A_1 \\
\vdots \\
A_i \\
\vdots \\
A_M \\
\vdots \\
A_1 \cup A_2 \\
\vdots \\
A_1 \cup \ldots \cup A_i \cup \ldots \cup A_M
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N
\end{array} \\
\left[\begin{array}{ccccc}
s_{11} & \ldots & s_{1j} & \ldots & s_{1N} \\
 & & \vdots & & \\
s_{i1} & \ldots & s_{ij} & \ldots & s_{iN} \\
 & & \vdots & & \\
s_{M1} & \ldots & s_{Mj} & \ldots & s_{MN} \\
 & & \vdots & & \\
s_{(M+1)1} & \ldots & s_{(M+1)j} & \ldots & s_{(M+1)N} \\
 & & \vdots & & \\
s_{(2^M-1)1} & \ldots & s_{(2^M-1)j} & \ldots & s_{(2^M-1)N}
\end{array}\right]
\end{array}
$$

See [Dezert Han Tacnet Carladous Yin 2016, Carladous 2017] for details

How to build $m(.) : 2^{\mathcal{A} \triangleq \{A_1, A_2, \ldots, A_M\}} \mapsto [0,1]$ from scores $\mathbf{S} \triangleq [S_{ij}]$?

Direct extension of BBA construction [Dezert Han Tacnet Carladous Yin 2016]

- Positive support of $X_i \in 2^{\mathcal{A}}$ based on all scores values of a criteria C_j

$$Sup_j(X_i) \triangleq \sum_{Y \in 2^{\mathcal{A}} | S_j(Y) \leqslant S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$Sup_j(X_i)$ measures how much X_i is better (higher) than other Y of $2^{\mathcal{A}}$

- Negative support of $X_i \in 2^{\mathcal{A}}$ based on all scores values of a criteria C_j

$$Inf_j(X_i) \triangleq - \sum_{Y \in 2^{\mathcal{A}} | S_j(Y) \geqslant S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$Inf_j(X_i)$ measures how much X_i is worse (lower) than other Y of $2^{\mathcal{A}}$

Belief function modeling

$$0 \leqslant \frac{Sup_j(X_i)}{X^j_{max}} \leqslant 1 - \frac{Inf_j(X_i)}{X^j_{min}} \leqslant 1 \Rightarrow \begin{cases} Bel_{ij}(X_i) \triangleq \dfrac{Sup_j(X_i)}{X^j_{max}}, \text{ with } X^j_{max} = \max_i Sup_j(X_i) \\ Bel_{ij}(\bar{X}_i) \triangleq \dfrac{Inf_j(X_i)}{X^j_{min}}, \text{ with } X^j_{min} = \min_i Inf_j(X_i) \end{cases}$$

Example 1

Five students $A_1, \ldots, A_5$ have to be ranked based on two criteria

- C_1 = long jump performance
- C_2 = collected funds for an animal protection project

The scores are given as follows

$$\mathbf{S} = \begin{array}{c} X_i \in 2^{\mathbf{A}} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \\ A_1 \cup A_2 \\ A_3 \cup A_4 \end{array} \begin{array}{c} C_1, w_1 \\ \left[\begin{array}{c} 3.7\,\mathrm{m} \\ 3.6\,\mathrm{m} \\ 3.8\,\mathrm{m} \\ 3.7\,\mathrm{m} \\ \varnothing \\ \varnothing \end{array}\right. \end{array} \begin{array}{c} C_2, w_2 \\ \left.\begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \\ 640\,€ \\ 600\,€ \\ 650\,€ \end{array}\right] \end{array}$$

Difficulties:

- Scores are given in different units and different scales
- Some scores values can be missing
- Criteria C_j do not have same weights of importance w_j (in general)

Example of non classical MCDM problem with BF-TOPSIS1

Step 1: BBA matrix construction

$$\mathbf{S} = \begin{array}{c} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \\ A_1 \cup A_2 \\ A_3 \cup A_4 \end{array} \begin{array}{c} \text{FE} \in 2^{\mathcal{A}} \quad C_1, w_1 \quad C_2, w_2 \\ \left[\begin{array}{cc} 3.7\,\text{m} & \varnothing \\ 3.6\,\text{m} & \varnothing \\ 3.8\,\text{m} & \varnothing \\ 3.7\,\text{m} & 640\,€ \\ \varnothing & 600\,€ \\ \varnothing & 650\,€ \end{array}\right] \end{array} \Rightarrow \mathbf{M} = \begin{array}{c} C_1, w_1 \quad C_2, w_2 \\ \left[\begin{array}{cc} (0.25, 0.25, 0.50) & (0, 0, 1) \\ (0, 1, 0) & (0, 0, 1) \\ (1, 0, 0) & (0, 0, 1) \\ (0.25, 0.25, 0.50) & (0.6667, 0.1111, 0.2222) \\ (0, 0, 1) & (0, 1, 0) \\ (0, 0, 1) & (1, 0, 0) \end{array}\right] \end{array}$$

Step 2: distances to ideal best and worst solutions

Focal elem.	$d_{BI}(m_{i1}, m^{best})$	$d_{BI}(m_{i1}, m^{worst})$	$d_{BI}(m_{i2}, m^{best})$	$d_{BI}(m_{i2}, m^{worst})$
A_1	0.6016	0.2652	0.7906	0.2041
A_3	0.8416	0	0.7906	0.2041
A_4	0	0.8416	0.7906	0.2041
A_5	0.6016	0.2652	0.2674	0.5791
$A_1 \cup A_2$	0.5401	0.3536	0.6770	0
$A_3 \cup A_4$	0.5401	0.3536	0	0.6770

Steps 3-5: weighted distances with $w_1 = 1/3$ and $w_2 = 2/3$, closeness and ranking

Focal elem.	$d^{best}(X_i)$	$d^{worst}(X_i)$	$C(X_i, X^{best})$	Ranking
A_1	0.7276	0.2245	0.2358	4
A_3	0.8076	0.1361	0.1442	6
A_4	0.5270	0.4166	0.4415	3
A_5	0.3788	0.4745	0.5561	2
$A_1 \cup A_2$	0.6314	0.1179	0.1573	5
$A_3 \cup A_4$	0.1800	0.5692	0.7597	1

A more concrete example of non classical MCDM problem

Application: Protecting housing areas against torrential floods

Presented in [Dezert Han Tacnet Carladous Yin 2016, Carladous 2017]

List of alternatives (possible actions to take)

- A_1= maintenance of check dams' series
- A_2= no maintenance, but build a sediment trap upstream
- A_3= make individual protections to limit damage on buildings

List of criteria

- C_1 (in €) = investment cost (in negative values)
- C_2 (in €) = risk reduction in 50 years between the current situation and expected situation with the chosen action
- C_3 (in {1,2,...,10}) = impact on environment
- C_4 (in m^2) = the land-use areas needed in privates

Score matrix (the higher is the score, the better is the proposition)

$$\mathbf{S} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_1 \cup A_2 \\ A_1 \cup A_3 \\ A_2 \cup A_3 \\ A_1 \cup A_2 \cup A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1, w_1 = 0.33 & C_2, w_2 = 0.33 & C_3, w_3 = 0.20 & C_4, w_4 = 0.14 \end{array} \\ \begin{bmatrix} -150000 & 100000 & 10 & 0 \\ -500000 & 200000 & 2 & -20000 \\ -550000 & 250000 & 10 & -5000 \\ -650000 & 230000 & 2 & -20000 \\ -700000 & 250000 & 10 & -5000 \\ -1050000 & 250000 & 2 & -25000 \\ -1200000 & 250000 & 2 & -25000 \end{bmatrix} \end{array}$$

Note: the scores are not cumulative in the same way for each criterion. For C_1 and C_4, the score of the disjunction of two alternatives is the sum of individual scores whereas it is not the case for C_2 and C_3.

A more concrete example of non classical MCDM problem

Here we apply BFTOPSIS1 method for its simplicity

Step 1: BBA construction from score matrix $\mathbf{S}$

$$\mathbf{M} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_1 \cup A_2 \\ A_1 \cup A_3 \\ A_2 \cup A_3 \\ A_1 \cup A_2 \cup A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1, w_1 & C_2, w_2 & C_3, w_3 & C_4, w_4 \end{array} \\ \begin{bmatrix} (1,0,0) & (0,1,0) & (1,0,0) & (1,0,0) \\ (0.44,0.10,0.46) & (0.45,0.28,0.27) & (0,1,0) & (0.10,0.67,0.23) \\ (0.37,0.13,0.50) & (1,0,0) & (1,0,0) & (0.70,0.07,0.23) \\ (0.27,0.21,0.52) & (0.73,0.10,0.17) & (0,1,0) & (0.10,0.67,0.23) \\ (0.23,0.26,0.51) & (1,0,0) & (1,0,0) & (0.70,0.07,0.23) \\ (0.04,0.75,0.21) & (1,0,0) & (0,1,0) & (0,1,0) \\ (0,1,0) & (1,0,0) & (0,1,0) & (0,1,0) \end{bmatrix} \end{array}$$

Steps 2-5: weighted distances with $w_1 = w_2 = 0.33$, $w_3 = 0.20$, $w_4 = 0.14$, closeness and ranking

Focal elem. X_i	$d^{best}(X_i)$	$d^{worst}(X_i)$	$C(X_i, X^{best})$	Ranking
A_1	0.3012	0.6116	0.6700	3
A_2	0.5668	0.3677	0.3935	6
A_3	0.1830	0.7483	0.8035	2
$A_1 \cup A_2$	0.4476	0.4901	0.5226	4
$A_1 \cup A_3$	0.1555	0.7775	0.8333	1
$A_2 \cup A_3$	0.5562	0.3614	0.3938	5
$A_1 \cup A_2 \cup A_3$	0.8328	0.2694	0.2444	7

Final ranking: best action(s) to take

$$(A_1 \cup A_3) \succ A_3 \succ A_1 \succ (A_1 \cup A_2) \succ (A_2 \cup A_3) \succ A_2 \succ (A_1 \cup A_2 \cup A_3)$$

Toolboxes for working with belief functions

- To start working with BF, we recommend Smets TBM MatLab codes that include many useful efficient functions based on Fast Möbius Transforms

  `http://iridia.ulb.ac.be/~psmets/`

- Main toolboxes for working with BF can be found from Belief Functions and Applications Society (www.bfasociety.org) wiki webpage at

  `http://bfaswiki.iut-lannion.fr/wiki/index.php/Toolboxes`

- Explanations for implementation of generalized belief functions can be found in

  A. Martin, Implementing general belief function framework with a practical codification for low complexity, in [DSmT books], Vol. 3, Chap 7, 2009.

- Implementation of fusion rules by sampling techniques (java package)
  `http://refereefunction.fredericdambreville.com`

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Jean Dezert Short Biography



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Jean Dezert was born in France on August 25, 1962. He got his Ph.D. from Paris XI Univ., Orsay, in 1990 in Automatic Control and Signal Processing. During 1986-1990 he was with the Syst. Dept. at The French Aerospace Lab (ONERA) and did research in multi-sensor multi-target tracking (MS-MTT). During 1991-1992, he was post-doc at ESE dept., UConn, USA under supervision of Prof. Bar-Shalom. During 1992-1993 he was teaching assistant in EE Dept, Orléans Univ., France. Since 1993, he is Senior Research Scientist and Maître de Recherches at ONERA. His research interests include estimation theory, information fusion, reasoning under uncertainty, and multi-criteria decision-making support. He has organized Fusion conference in Paris in 2000 and has been TPC member of Fusion 2000-2017 conferences. He served as ISIF 2016 President (`www.isif.org`). Dr. Dezert published more than 150 papers in conferences and journals, and edited four books on Dezert-Smarandache Theory (DSmT) with Prof. Smarandache.

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