BFAS school
Practical session 1

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1 Exercises

1. Represent the uncertainty about the outcome of the Ellsberg's experiment, using a mass function on a suitable frame. Compute the corresponding belief, plausibility and commonality functions.

2. Let $\Omega = \{a, b\}$, and let m and m' be the following mass functions on Ω,
$$m = \{a\}^{\alpha} \oplus \{b\}^{\beta}, \quad m' = \{a\}^{\alpha'} \oplus \{b\}^{\beta'}.$$
   (a) Compute m and m'.
   (b) Compute $m \oplus m'$.
   (c) Compute $m \textcircled{\wedge} m'$.

3. Let $[U, V]$ be a random interval. Compute $Bel^{\Theta}((-\infty, u])$ and $Pl^{\Theta}((-\infty, u])$ for any $u \in \mathbb{R}$, using the cdfs of U and V.

4. Let $Y_1, \ldots, Y_n, Z$ be an iid sample from a uniform distribution in $[0, \theta]$. Assume that $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is observed and Z is not yet observed.
   (a) Describe the likelihood-based belief function on θ.
   (b) Describe the predictive belief function on Z.

2 Solutions

1. A suitable frame is $\Omega = \{R, B, Y\}$.

A	$\emptyset$	$\{R\}$	$\{B\}$	$\{Y\}$	$\{R,B\}$	$\{R,Y\}$	$\{B,Y\}$	$\{R,B,Y\}$
$m(A)$	0	1/3	0	0	0	0	2/3	0
$Bel(A)$	0	1/3	0	0	1/3	1/3	2/3	1
$Pl(A)$	0	1/3	2/3	2/3	1	1	2/3	1
$Q(A)$	1	1/3	2/3	2/3	0	0	2/3	0

2. (a) We have

$$m(\{a\}) = \frac{\beta(1-\alpha)}{\alpha+\beta-\alpha\beta}, \quad m(\{b\}) = \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta},$$
$$m(\{a,b\}) = \frac{\alpha\beta}{\alpha+\beta-\alpha\beta}.$$

A similar expression is obtained for m' by replacing α and β by α' and β'.

(b) $m \oplus m' = \{a\}^{\alpha\alpha'}\{b\}^{\beta\beta'}$. Consequently,

$$m(\{a\}) = \frac{\beta\beta'(1-\alpha\alpha')}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'}, \quad m(\{b\}) = \frac{\alpha\alpha'(1-\beta\beta')}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'},$$
$$m(\{a,b\}) = \frac{\alpha\alpha'\beta\beta'}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'}.$$

(c) $m \textcircled{\wedge} m' = \{a\}^{\min(\alpha,\alpha')}\{b\}^{\min(\beta,\beta')}$. The masses can easily be computed as above.

3. For all $u \in \mathbb{R}$, we have:

$$Bel((-\infty,u]) = \mu([U,V] \subseteq (-\infty,u]) = \mu(V \leq u) = F_V(u),$$

where F_V is the cumulative distribution function (cdf) of V, and

$$Pl((-\infty,u]) = \mu([U,V] \cap (-\infty,u] \neq \emptyset) = \mu(U \leq u) = F_U(u).$$

4. (a) The likelihood function is

$$L_{\mathbf{y}}(\theta) = \theta^{-n}\mathbb{1}_{[y_{(n)},+\infty)}(\theta),$$

where $y_{(n)} = \max_{1\leq i\leq n} y_i$, and the contour function is

$$pl_{\mathbf{y}}(\theta) = \left(\frac{y_{(n)}}{\theta}\right)^n \mathbb{1}_{[y_{(n)},+\infty)}(\theta).$$

We note that, the contour function being unimodal and upper-semicontinous, the focal sets $\Gamma_{\mathbf{y}}(s)$ are close intervals $[\widehat{\theta}_{\mathbf{y}*}(s), \widehat{\theta}^*_{\mathbf{y}}(s)]$, with $\widehat{\theta}_{\mathbf{y}*}(s) = y_{(n)}$ and $\widehat{\theta}^*_{\mathbf{y}}(s) = y_{(n)}s^{-1/n}$ for all $s \in [0,1]$. Consequently, the belief function $Bel^{\Theta}_{\mathbf{y}}$ is induced by the random closed interval $[y_{(n)}, y_{(n)}S^{-1/n}]$, with $S \sim \mathcal{U}([0,1])$.

(b) As $F_\theta(z) = z/\theta$ for all $0 \leq z \leq \theta$, we can write $Z = \theta W$ with $W \sim \mathcal{U}([0,1])$. As function $\varphi(\theta, W) = \theta W$ is continuous in θ, each focal set of $Bel^{\mathbb{Z}}_{\mathbf{y}}$ is an interval

$$\varphi(\Gamma_{\mathbf{y}}(s), w) = [y_{(n)}w, y_{(n)}s^{-1/n}w],$$

so that $Bel^{\mathbb{Z}}_{\mathbf{y}}$ is induced by the random interval

$$[\widehat{Z}_{\mathbf{y}*}, \widehat{Z}^*_{\mathbf{y}}] = [y_{(n)}W, y_{(n)}S^{-1/n}W].$$