UNCERTAINTY THEORIES: A UNIFIED VIEW

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Outline

1. Variability vs ignorance
2. Set-valued representations of ignorance, vs. representations of real sets
3. Blending set-valued and probabilistic representations : uncertainty theories
4. Comparison of uncertainty theories on
    1. Practical representations
    2. Information content
    3. Conditioning
    4. Combination

Origins of uncertainty

- The variability of observed natural phenomena : **randomness.**
  - Coins, dice…: what about the outcome of the next throw?
- The lack of information: **incompleteness**
  - because of information is often lacking, knowledge about issues of interest is generally not perfect.
- Conflicting testimonies or reports: **inconsistency**
  - The more sources, the more likely the inconsistency

Example

- **Frequentist**: daily quantity of rain in Toulouse
  - Represents variability: it may change every day
  - It is objective: can be estimated through statistical data
- **Incomplete information** : Birth date of Brazilian President
  - It is not a variable: it is a constant!
  - Information is incomplete
  - It subjective: Most may have a rough idea (an interval), a few know precisely, some have no idea.
  - Statistics on birth dates of other presidents do not help much.

Knowledge vs. evidence

- There are two kinds of information that help us make decisions in the course of actions:
  - Generic knowledge:
    - pertains to a population of observables (e.g. statistical knowledge)
    - Describes a general trend based on objective data
    - Tainted with exceptions
    - Deals with observed frequencies or ideas of typicality
  - Singular evidence:
    - Consists of direct information about the current world.
    - pertains to a single situation
    - Can be unreliable, uncertain (e.g. unreliable testimony)

The roles of probability

Probability theory is generally used for representing two types of phenomena:

1. **Randomness:** capturing variability through repeated observations.
2. **Belief:** describes a person's opinion on the occurrence of a singular event.

As opposed to frequentist probability, subjective probability that models unreliable evidence is not necessarily related to statistics.

Constructing beliefs

- Belief in the occurrence of a particular event may derive from its statistical probability: the **Hacking principle**:
  - Generic knowledge = probability distribution P
  - $\text{belief}_{\text{NOW}}(A) = \text{Freq}_{\text{POPULATION}}(A)$: equating belief and frequency
- Beliefs can be directly elicited as subjective probabilities **of singular events** with no frequentist flavor
  - frequencies may not be available nor known
  - non repeatable events.
- But *a single subjective probability distribution cannot distinguish between uncertainty due to variability and uncertainty due to lack of knowledge*

SUBJECTIVE PROBABILITIES
(Bruno de Finetti, 1935)

- p_i = *belief degree* of an agent on the (next) occurrence of s_i
- measured as the price of a lottery ticket with reward 1 € if state is s_i in a betting game
- **Rules of the game**:
  - gambler proposes a price p_i
  - banker and gambler exchange roles if banker finds price p_i is too low
- **Why a belief state is a single distribution** ($\sum_j p_j = 1$):
  - Assume player buys all lottery tickets i = 1, m.
  - If state s_j is observed, the gambler gain is $1 - \sum_j p_j$
  - and $\sum_j p_j - 1$ for the banker
  - *if* $\sum p_j > 1$ gambler *always loses money ;*
  - *if* $\sum p_j < 1$ *banker exchanges roles with* gambler

Using a single probability distribution to represent incomplete information is not entirely satisfactory:

The betting behavior setting of Bayesian subjective probability enforces a representation of partial ignorance based on single probability distributions.

1. **Ambiguity :** In the absence of information, how can a uniform diistribution tell pure randomness and ignorance apart ?
2. **Instability :** A uniform prior on $x \in [a, b]$ induces a non-uniform prior on $f(x) \in [f(a), f(b)]$ if f is increasing and non-affine.
3. **Empirical refutation**: When information is missing, decision-makers do not always choose according to a single subjective probability (Ellsberg paradox).

Motivation for going beyond probability

- Distinguish between uncertainty due to variability from uncertainty due to lack of knowledge or missing information.
- **The main tools to representing uncertainty are**
  - **Probability distributions :** good for expressing variability, but information demanding
  - **Sets**: good for representing incomplete information, but often crude representation of uncertainty
- *Find representations that allow for both aspects of uncertainty*.

Set-Valued Representations of Partial Knowledge

- An ill-known quantity x is represented as a disjunctive set, i.e. a subset E of *mutually exclusive values*, one of which is the real one.
- Pieces of information of the form $x \in E$
  - **Intervals** E = [a, b]: good for representing incomplete numerical information
  - **Classical Logic**: good for representing incomplete symbolic (Boolean) information

    E = Models of a wff ϕ stated as true.

This kind of information is subjective (epistemic set)

BOOLEAN POSSIBILITY THEORY

Natural set functions under incomplete information:

If all we know is that $x \in E$ *then*

- Event A is possible if $A \cap E \neq \emptyset$

(logical consistency)

$\Pi(A) = 1$, and 0 otherwise

- Event A is sure if $E \subseteq A$

(logical deduction)

$N(A) = 1$, and 0 otherwise

This is a simple modal logic (KD45)

$$N(A) = 1 - \Pi(A^c)$$

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B));$$

$$N(A \cap B) = \min(N(A), N(B)).$$

SETS and SETS

- Do not mix up
  - **A set-valued variable X**: the set of languages a person can speak (A = {English (and) French}) *it is a conjunction of values, and a real set.*

    X = A is precise
  - **An ill-known point-valued variable x** : the language this person is speaking now

    (E = {English (exor) French}) : *it is a disjunction of values, and an epistemic set (a piece of information in the head of an agent).*

    $x \in E$ *is imprecise.*

Find a representation of uncertainty due to incompleteness

- *More informative than sets*
- *Less demanding than single probability distributions*
- *Explicitly allows for missing information*
- *Allows for addressing the same problems as probability.*

Blending intervals and probability

- Representations that may account for both variability and incomplete knowledge must combine probability and sets.
  - Sets of probabilities : imprecise probability theory
  - Random(ised) sets : Dempster-Shafer theory
  - Fuzzy sets: numerical possibility theory
- Each event has a degree of belief (certainty) and a degree of plausibility, instead of a single degree of probability

A GENERAL SETTING FOR REPRESENTING GRADED CERTAINTY AND PLAUSIBILITY

- 2 conjugate set-functions Pl and Cr generalizing probability P, possibility Π, and necessity N.
- **Conventions** :
  - Pl(A) = 0 "impossible" ; Cr(A) = 1 "certain"
  - Pl(A) =1 ; Cr(A) = 0 "ignorance" **(no information)**
  - Pl(A) - Cr(A) quantifies ignorance about A
- **Postulates**
  - Cr and Pl are monotonic under inclusion (= capacities).
  - $Cr(A) \leq Pl(A)$ "certain implies plausible"
  - $Pl(A) = 1 - Cr(A^c)$ duality certain/plausible
  - If Pl = Cr then it is P.

Possibility Theory

(Shackle, 1961, Zadeh, 1978)

- A piece of incomplete information "$x \in E$" admits of *degrees* of possibility: E is a (normalized) fuzzy set.
- $\mu_E(s)$ = Possibility(x = s) = $\pi_x(s)$
- **Conventions:**

  $\forall s, \pi_x(s)$ is the degree of plausibility of x = s

  $\pi_x(s) = 0$ iff x = s is impossible, totally surprising

  $\pi_x(s) = 1$ iff x = s is normal, fully plausible, unsurprising (but no certainty)

A pioneer of possibility theory

- In the 1950's, **G.L.S. Shackle** called "degree of potential surprize" of an event its degree of impossibility = $1 - \Pi(A)$.

- Potential surprize is valued on a disbelief scale, namely a positive interval of the form $[0, y^*]$, where y^* denotes the absolute rejection of the event to which it is assigned, and 0 means that nothing opposes to the occurrence of A.

- The degree of surprize of an event is the degree of surprize of its least surprizing realization.
- He introduces a notion of conditional possibility

Improving expressivity of incomplete information representations

- *What about the birth date of the president?*
- **partial ignorance with ordinal preferences :** May have reasons to believe that 1933 > 1932 ≡ 1934 > 1931 ≡ 1935 > 1930 > 1936 > 1929
- **Linguistic information** described by fuzzy sets: " he is old " : membership μ_{OLD} is ionterpreted as a possibility distribution on possible birth dates.

- **Nested intervals** $E_1, E_2, \ldots E_n$ **with confidence levels** $N(E_i) = a_i$:
  - $\pi(x) = \min_{i=1,\ldots n} \max(\mu_{Ei}(x), 1 - a_i)$

POSSIBILITY AND NECESSITY OF AN EVENT

How confident are we that $x \in A \subset S$? (*an event A occurs*)
given a possibility distribution on S

- $\Pi(A) = \max_{s \in A} \pi(s)$:
  to what extent A is consistent with π
  (= some $x \in A$ is possible)
  The degree of possibility *that* $x \in A$
- $N(A) = 1 - \Pi(A^c) = \min_{s \notin A} 1 - \pi(s)$:
  to what extent no element outside A is possible
  = to what extent π implies A
  The degree of certainty (necessity) that $x \in A$

Basic properties

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B));$$

$$N(A \cap B) = \min(N(A), N(B)).$$

Mind that most of the time :

$$\Pi(A \cap B) < \min(\Pi(A), \Pi(B));$$

$$N(A \cup B) > \max(N(A), N(B)$$

Example: Total ignorance on A and B = A^c

Corollary $N(A) > 0 \Rightarrow \Pi(A) = 1$

Qualitative vs. quantitative possibility theories

- **Qualitative**:
  - **comparative**: A complete pre-ordering $\geq_\pi$ on U A well-ordered partition of U: E1 > E2 > … > En
  - **absolute:** $\pi_x(s) \in L$ = finite chain, complete lattice...
- **Quantitative**: $\pi_x(s) \in [0, 1]$, integers...

One must indicate where the numbers come from.

All theories agree on the fundamental maxitivity axiom

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$$

Theories diverge on the conditioning operation

Imprecise probability theory

- A state of information is represented by a family $\mathcal{P}$ of probability distributions over a set X.
- To each event A is attached a probability interval $[P_*(A), P^*(A)]$ such that
  - $P_*(A) = \inf\{P(A), P \in \mathcal{P}\}$
  - $P^*(A) = \sup\{P(A), P \in \mathcal{P}\} = 1 - P_*(A^c)$
- Usually $\mathcal{P}$ is strictly contained in $\{P(A), P \geq P_*\}$
- *For instance: incomplete knowledge of an objective probabilistic model :* $\exists P \in \mathcal{P}$.

Subjectivist view (Peter Walley)

- $P_{low}(A)$ is the highest acceptable price for buying a bet on singular event A winning 1 euro if A occurs
- $P^{high}(A) = 1 - P_{low}(A^c)$ is the least acceptable price for selling this bet.
- **Rationality** conditions:
  - **No sure loss** : $\{P \geq P_{low}\}$ not empty
  - **Coherence**: $P_*(A) = \inf\{P(A), P \geq P_{low}\} = P_{low}(A)$
- *A theory that handles convex probability sets*
- Convex probability sets are actually characterized by lower expectations of real-valued functions (gambles), not just events.

Random sets

- A probability distribution *m* on the family of non-empty subsets of a set S.
- A positive weighting of non-empty subsets: mathematically, **a random set** :

$$\sum_{E \in \mathcal{F}} m(E) = 1$$

- m : *mass function*.
- *focal sets* : $E \in \mathcal{F}$ with $m(E) > 0$: *disjunctive or conjunctive ???*

Random disjunctive sets

- A focal set is an epistemic state ($x \in E$)
- ($\mathcal{F}$, m) is a randomized incomplete information
- m(E) = probability that the most precise description of the available information is of the form "$x \in E$"

  = *probability (only knowing "$x \in E$" and nothing else)*

  – It is the portion of probability mass hanging over elements of E without being allocated.
- **DO NOT MIX UP m(E) and P(E)**

Basic set functions from random sets

- **degree of certainty** (belief) :
  - $\mathrm{Bel}(A) = \sum_{E_i \subseteq A,\, E_i \neq \emptyset} m(E_i)$
  - total mass of information implying the occurrence of A
  - *(probability of provability)*
- **degree of plausibility** :
  - $\mathrm{Pl}(A) = \sum_{E_i \cap A \neq \emptyset} m(E_i) = 1 - \mathrm{Bel}(A^c) \geq \mathrm{Bel}(A)$
  - total mass of information consistent with A
  - *(probability of consistency)*

PARTICULAR CASES

- INCOMPLETE INFORMATION:

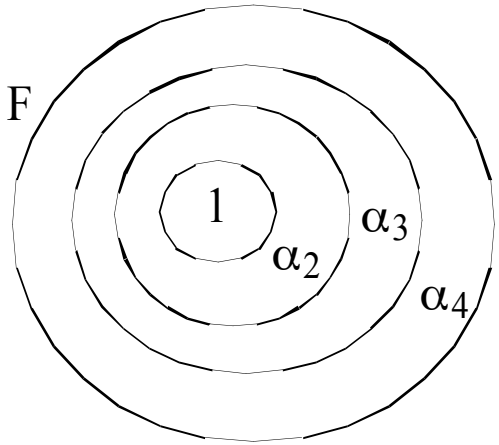
  $m(E) = 1, m(A) = 0,\ A \neq E$

- *TOTAL IGNORANCE :* $m(S) = 1$:
  - *For all* $A \neq S, \emptyset$, *Bel(A) = 0, Pl(A) = 1*
- PROBABILITY: if $\forall i, E_i$ = singleton $\{s_i\}$ (hence disjoint focal sets)
  - Then, *for all* A, $Bel(A) = Pl(A) = P(A)$
  - *Hence precise + scattered information*
- POSSIBILITY THEORY : the consonant case

  $E_1 \subseteq E_2 \subseteq E_3 \ldots \subseteq E_n$: imprecise and coherent information
  - iff $Pl(A \cup B) = \max(Pl(A), Pl(B))$, possibility measure
  - iff $Bel(A \cap B) = \min(Bel(A), Bel(B))$,

    necessity measure = consonant belief function

From possibility to random sets



possibility levels
$1 > \alpha_2 > \alpha_3 > \ldots > \alpha_n$

- Let $m_i = \alpha_i - \alpha_{i+1}$ then $m_1 + \ldots + m_n = 1$,
  with focal sets = cuts $A_i = \{s, \pi(s) \geq \alpha_i\}$

  A basic probability assignment (SHAFER)
- $\pi(s) = \sum_{i:\, s \in Fi} m_i$ (one point-coverage function) = $Pl(\{s\})$.
- ***Only in the consonant case can m be recalculated from π***
- *$Bel(A) = \sum_{Fi \subseteq A} m_i = N(A)$; $Pl(A) = \Pi(A)$*

Random disjunctive sets vs. imprecise probabilities

- The set $\mathcal{P}_{bel} = \{P \geq Bel\}$ is coherent: Bel is a special case of lower probability
- Bel is ∞-monotone (super-additive at any order)
- The solution m to the set of equations $\forall A \subseteq X$

$$g(A) = \sum_{E_i \subseteq A, E_i \neq \emptyset} m(E_i)$$

is unique (Moebius transform)

– ***However m is positive iff g is a belief function***

Dempster original model

- Indirect information (induced from a probability space).
- What we know about a random variable x with range S, based on a sample space $(\Omega, \mathcal{A}, P)$ and a multimapping Γ from Ω to S (Dempster):
- *The meaning of the multimapping Γ from Ω to S* :
  - if we observe ω in Ω then all we know is $x(\omega) \in \Gamma(\omega)$
- $m(\Gamma(\omega)) = P(\{\omega\})$ $\forall$ ω in Ω (finite case.)
- Then a belief function is a lower probability.

Dempster vs. Shafer-Smets

- A disjunctive random set can represent
  - *Uncertain singular evidence* (unreliable testimonies): m(E) = subjective probability pertaining to the truth of testimony E.
    - Degrees of belief directly modelled by Bel : no appeal to an underlying probability.

  (Shafer, 1976 book; Smets)

  - *Imprecise statistical evidence*: m(E) = frequency of imprecise observations of the form E and Bel(E) is a lower probability
    - A multiple-valued mapping from a probability space to a space of interest representing an ill-known random variable.
    - Here, belief functions are explicitly viewed as lower probabilities

  (Dempster intuition)
- *In all cases E is a set of mutually exclusive values and does not represent a real set-valued entity*

Canonical examples

- **Statistical probabilities** : Frequentist modelling of a collection of incomplete observations (imprecise statistics) : incomplete generic information
- **Uncertain singular information:**
  - **Unreliable testimonies** (Shafer's book) : unreliable singular information
  - **Unreliable sensors** : the quality of the information depends on the ill-known sensor state.

Example of uncertain evidence : **Unreliable testimony** (SHAFER-SMETS VIEW)

- « John tells me the president is between 60 and 70 years old, but there is some chance (*subjective* probability p) he does not know and makes it up».
  - E =[60, 70]; Prob(Knowing "x∈ E =[60, 70]") = 1 − p.
  - With probability p, John invents the info, so *we know nothing (Note that this is different from a lie)*.
- We get a *simple support belief function :*

$$m(E) = 1 - p \quad \text{and} \quad m(S) = p$$

- Equivalent to a possibility distribution
  - $\pi(s) = 1$ if $x \in E$ and $\pi(s) = p$ otherwise.
- *If John may lie* (probability q): $m(E) = (1 - p)(1 - q)$, $m(E^c) = (1 - p)q$.

Example of generic belief function: imprecise observations in an opinion poll

- **Question** : who is your preferred candidate in $C = \{a, b, c, d, e, f\}$???
  - **To a population** $\Omega = \{1, \ldots, i, \ldots, n\}$ of n persons.
  - **Imprecise responses r =** « $x(i) \in E_i$ » **are allowed**
  - No opinion ($r = C$) ; « left wing » $r = \{a, b, c\}$;
  - « right wing » $r = \{d, e, f\}$;
  - a moderate candidate : $r = \{c, d\}$
- **Definition of mass function**:
  - $m(E) = card(\{i, E_i = E\})/n$
  - = Proportion of imprecise responses « $x(i) \in E$ »

- *The probability that a candidate in subset* $A \subseteq C$ *is elected is imprecise :*

$$\text{Bel}(A) \leq P(A) \leq \text{Pl}(A)$$

- **There is a fuzzy set F of potential winners**:

$$\mu_F(x) = \sum_{x \in E} m(E) = \text{Pl}(\{x\}) \text{ (contour function)}$$

- $\mu_F(x)$ is an upper bound of the probability that x is elected. It gathers responses of those who *did not give up voting* for x
- Bel({x}) gathers responses of those who claim they will vote for x and no one else.

Example of conjunctive random sets

Experiment on linguistic capabilities of people :

- **Question** to **a population** $\Omega = \{1, \ldots, i, \ldots, n\}$ of n persons: which languages can you speak ?
- **Answers** : Subsets in $\mathcal{L} = \{$Basque, Chinese, Dutch, English, French,.…$\}$?
- m(E) = % people who speak *exactly* all languages in E (and not other ones)
- Prob(x speaks A) $= \sum\{m(E) : A \subseteq E\} = Q(A)$: commonality function in belief function theory
- **Example**: « x speaks English » means « at least English »
- The belief function is not meaningful here while the commonality makes sense, contrary to the disjunctive set case.

POSSIBILITY AS UPPER PROBABILITY

- Given a numerical possibility distribution π, define
  $\mathsf{P}(\pi) = \{P \mid P(A) \leq \Pi(A) \text{ for all } A\}$
  - Then, Π and N can be recovered
    - $\Pi(A) = \sup \{P(A) \mid P \in \mathsf{P}(\pi)\}$;
    - $N(A) = \inf \{P(A) \mid P \in \mathsf{P}(\pi)\}$
  - So π is a faithful representation of a special family of probability measures
- Likewise for belief functions : $\mathsf{P}(m) = \{P \mid P(A) \leq Pl(A), \forall A\}$
  - $Pl(A) = \sup \{P(A) \mid P \in \mathsf{P}(m)\}$;
  - $Bel(A) = \inf \{P(A) \mid P \in \mathsf{P}(m)\}$
- But this view makes sense for belief functions
  - Representing statistical knowledge
  - Representing maximal buying prices as per Walley

LIKELIHOOD FUNCTIONS

- **Likelihood functions** $\lambda(x) = P(A|\, x)$ behave like possibility distributions when there is no prior on x, and $\lambda(x)$ is used as the likekihood of x.
- It holds that $\lambda(B) = P(A|\, B) \leq \max_{x \in B} P(A|\, x)$
- If $P(A|\, B) = \lambda(B)$ then λ should be set-monotonic:

  $\{x\} \subseteq B$ implies $\lambda(x) \leq \lambda(B)$

  It implies $\lambda(B) = \max_{x \in B} \lambda(x)$

Maximum likelihood principle is possibility theory

- The classical coin example: θ is the unknown probability of "heads"
- Within n experiments: k heads, n-k tails
- P(k heads, n-k tails | θ) = $\theta^k \cdot (1-\theta)^{n-k}$ is

the degree of possibility $\pi(\theta)$ that the probability of "head" is θ.

In the absence of other information the best choice is the one that maximizes $\pi(\theta)$, $\theta \in [0, 1]$

It yields $\theta = k/n$.

LANDSCAPE OF UNCERTAINTY THEORIES

BAYESIAN/STATISTICAL PROBABILITY: the language of *unique* probability distributions (*Randomized points)*

UPPER-LOWER PROBABILITIES : the language of *disjunctive convex sets of probabilities, and lower expectations*

SHAFER-SMETS BELIEF FUNCTIONS: The language of Moebius masses (*Random disjunctive sets)*

QUANTITATIVE POSSIBILITY THEORY : The language of possibility distributions (*Fuzzy (nested disjunctive) sets)*

BOOLEAN POSSIBILITY THEORY (modal logic KD45) :
The language of *Disjunctive sets*

Practical representation issues

- Lower probabilities are difficult to represent ($2^{|S|}$ values): The corresponding family is a polyhedron with potentially |S|! vertices.
- Finite random sets are simpler but potentially $2^{|S|}$ values
- Possibility measures are simple (|S| values) but sometimes not expressive enough.
- ***There is a need for simple and more expressive representations of imprecise probabilities or random sets.***

Practical representations

- Fuzzy intervals (continuous consonant bf)
- Probability intervals
- Probability boxes (continuous random intervals with comonotonic bounds)
- Generalized p-boxes
- Clouds

Some are special random sets some not.

From confidence sets to possibility distributions

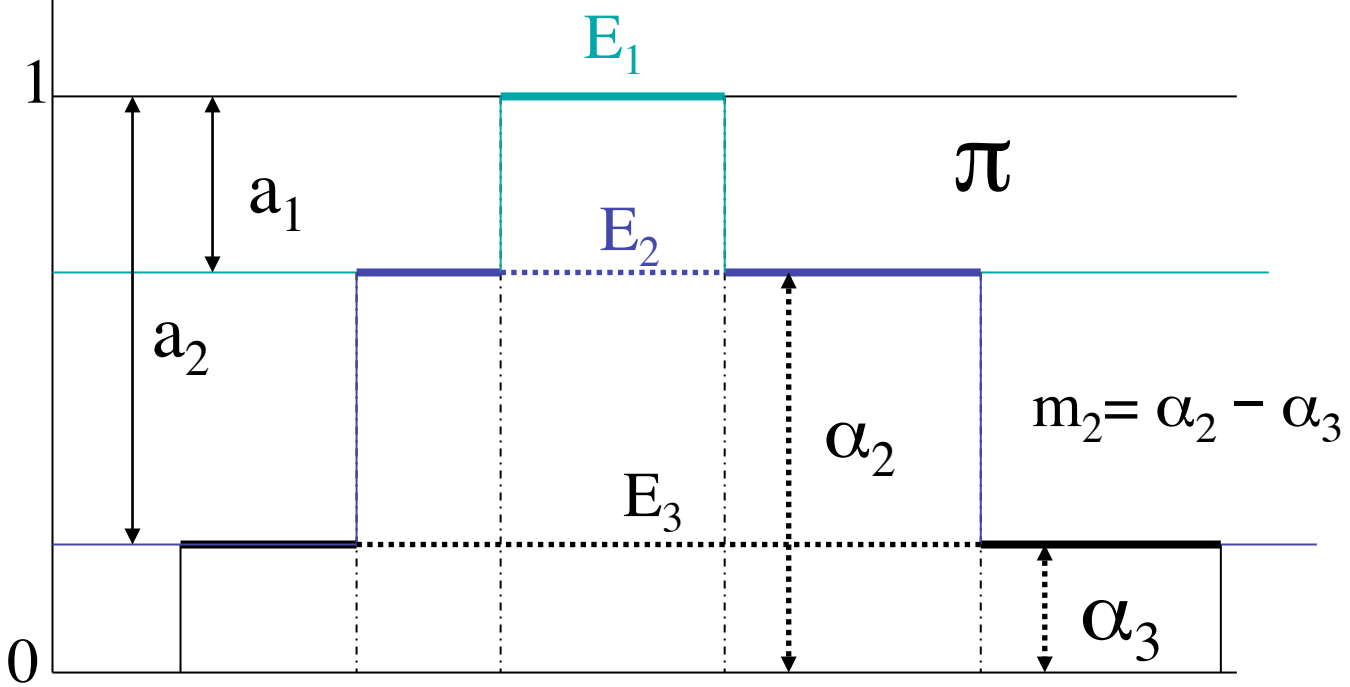
- Let $E_1, E_2, \ldots E_n$ be a nested family of sets
- A set of confidence levels $a_1, a_2, \ldots a_n$ in $[0, 1]$
- Consider the set of probabilities

$$\mathcal{P} = \{P, P(E_i) \geq a_i, \text{ for } i = 1, \ldots n\}$$

- Then $\mathcal{P}$ is representable by means of a possibility measure with distribution

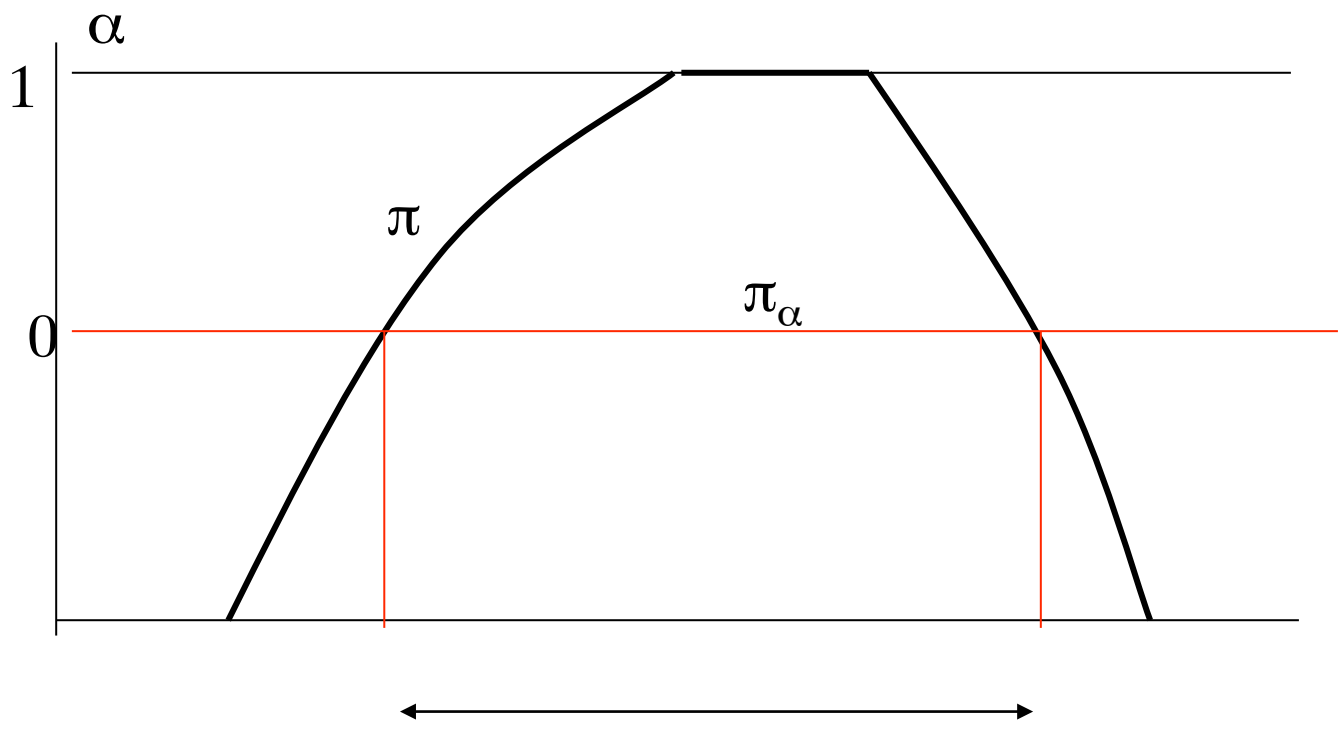
$$\pi(x) = \min_{i = 1, \ldots n} \max (\mu_{Ei}(x), 1 - a_i)$$

POSSIBILITY DISTRIBUTION INDUCED BY EXPERT CONFIDENCE INTERVALS



A possibility distribution can be obtained from any family of nested confidence sets :

$P(A_\alpha) \geq 1 - \alpha,\ \alpha \in (0, 1]$



FUZZY INTERVAL: $N(\pi_\alpha) = 1 - \alpha$

Possibilistic view of probabilistic inequalities

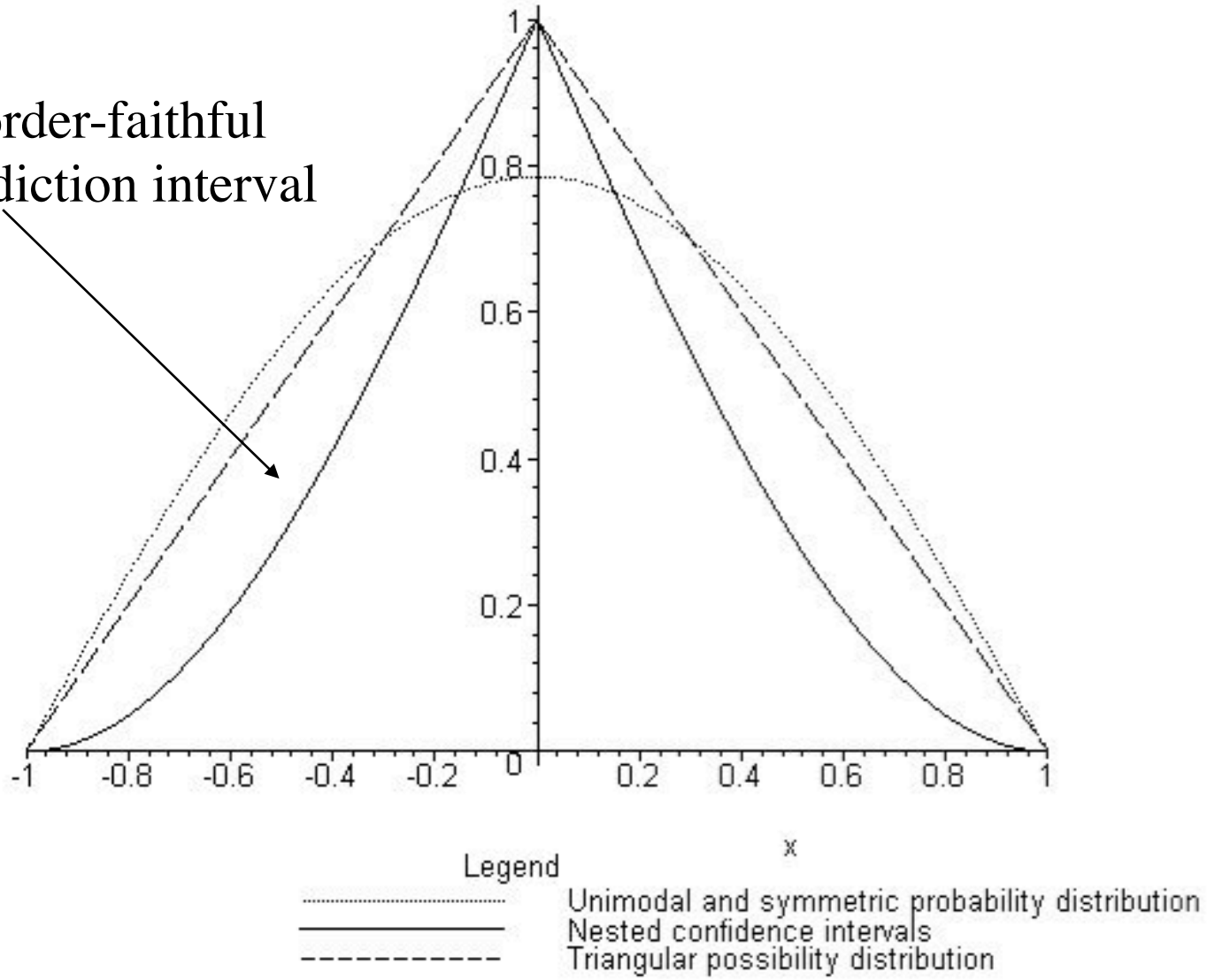
- Probabilistic inequalities can be used for knowledge representation
  - Chebyshev inequality defines a possibility distribution that dominates *any* density with given mean and variance:

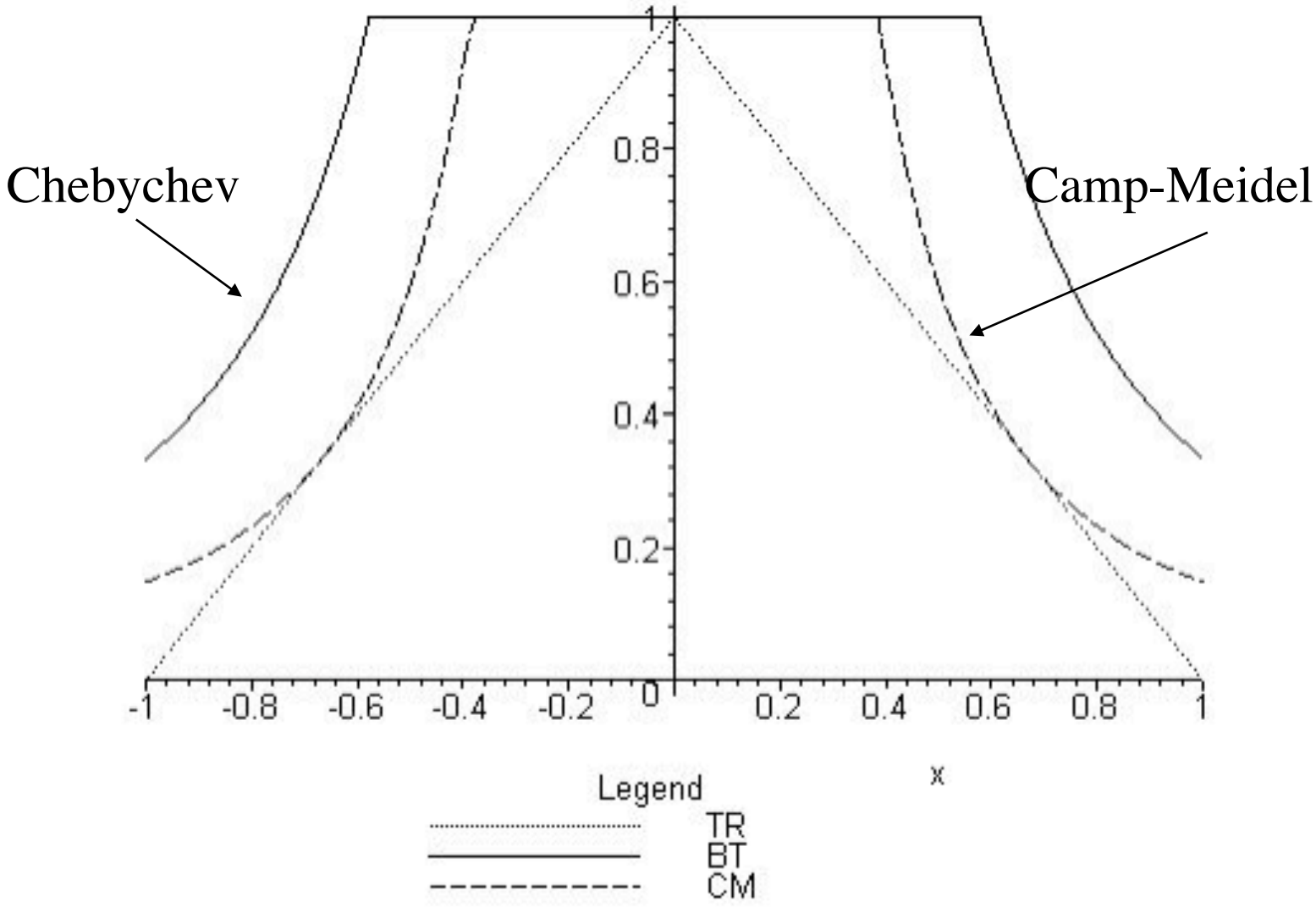
$P(V \in [x^{mean} - k\sigma, x^{mean} + k\sigma]) \geq 1 - 1/k^2$ *is equivalent to writing*

$\pi(x^{mean} - k\sigma) = \pi(x^{mean} + k\sigma) = 1/k^2$

  - A triangular fuzzy number (TFN) defines a possibility distribution that dominates *any* unimodal density with the same mode and bounded support as the TFN.

Optimal order-faithful
fuzzy prediction interval




Chebychev
Camp-Meidel
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
x
Legend
TR
BT
CM

Optimal order-faithful fuzzy prediction intervals

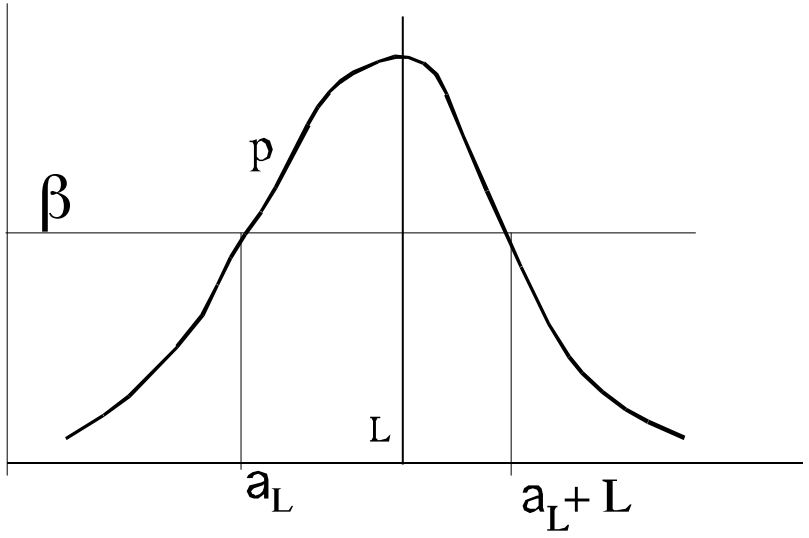
- the interval I_L of fixed length L with maximal probability is of the form

$\{x, p(x) \geq \beta\} = [a_L, a_L + L]$

- The most narrow prediction interval with probability α is of the form $\{x, p(x) \geq \beta\}$
- So the most natural possibilistic counterpart of p is when

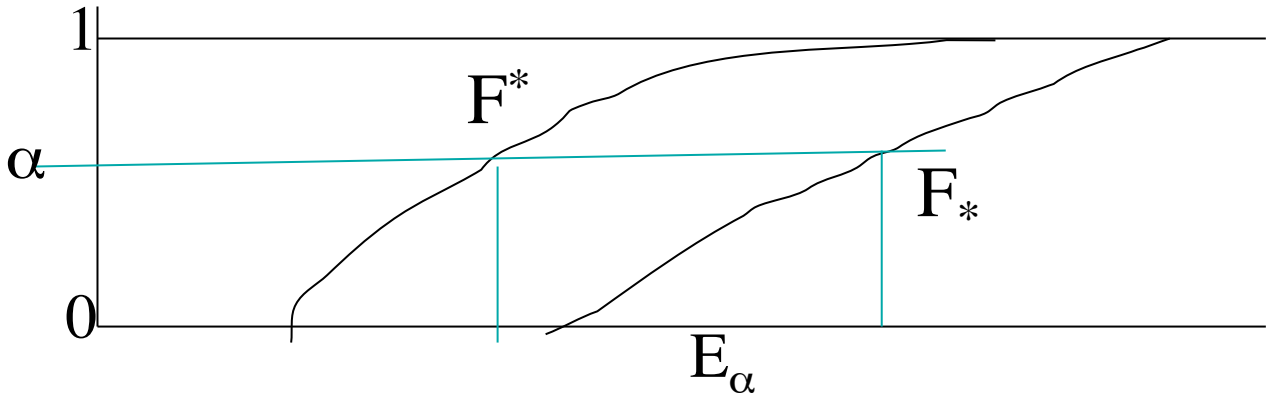
$\pi^*(a_L) = \pi^*(a_L + L) =$

$1 - P(I_L = \{x, p(x) \geq \beta\})$.



Probability boxes

- A set $\mathcal{P} = \{P: F^* \geq P \geq F_*\}$ induced by two cumulative disribution functions is called a **probability box (p-box)**,
- **A p-box is a special random interval (continuous belief function) whose upper and bounds induce the same ordering.**
- **A fuzzy interval induces a p-box $\mathcal{P}$**

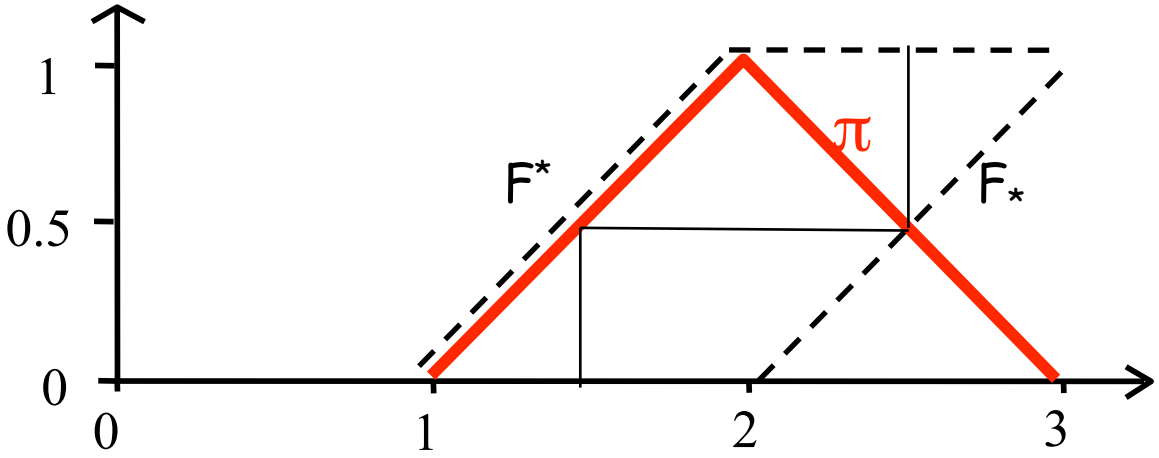


Probability boxes from possibility distributions

- *Representing families of probabilities by fuzzy intervals is more precise than with the corresponding pairs of PDFs:*
  - $F^*(a) = \Pi_M((-\infty, a]) = M(a)$ if $a \leq m$
    $= 1$ otherwise.
  - $F_*(a) = N_M((-\infty, a]) = 0$ if $a < m^*$
    $= 1 - \lim_{x \downarrow a} M(x)$ otherwise
- $\mathcal{P}(\pi)$ is a proper subset of $\mathcal{P}$
  - Not all P in $\{P: F^* \geq P \geq F_*\}$ are such that $\Pi \geq P$

P-boxes vs. fuzzy intervals

A triangular fuzzy number with support [1, 3] and mode 2.
Let P be defined by P({1.5})=P({2.5})=0.5.
Then $F_* < F < F^* \; P \notin P(\Pi)$ since
$P(\{1.5, 2.5\}) = 1 > \Pi(\{1.5, 2.5\}) = 0.5$



Generalized cumulative distributions

- A Cumulative distribution function F of a probability function P can be viewed as a possibility distribution dominating P

  $$\sup\{F(x), x \in A\} \geq P(A)$$

- Choosing any order relation $\leq_R$

  $F_R(x) = P(\{y \leq_R x\})$ also induces a possibility distribution dominating P

Generalized p-boxes

- The notion of cumulative distribution depends on an ordering on the space: $F_R(x) = P(X \leq_R x)$
- A generalized probability box is a pair of cumulative functions (F_R^*, F_{R*}) associated to the same order relation.

  $$\mathcal{P} = \{P: F_R^* \geq P \geq F_{R*}\}$$
- Consider $y \leq_R x$ iff $|y - a| \geq |x - a|$ (distance to a value)
- Then $\pi(y) = F_R^*(y) \geq \delta(y) = F_{R*}(y)$
- It comes down to considering nested confidence intervals $E_1, E_2, \ldots E_n$ each with two probability bounds α_i and β_i such that

  $$\mathcal{P} = \{ \alpha_i \leq P(E_i) \leq \beta_i \text{ for } i = 1, \ldots, n\}$$

Generalized p-boxes

- It comes down to two possibility distributions

  π (from $\alpha_i \leq P(E_i)$) and π_c (from $P(E_i) \leq \beta_i$)
- Distributions π and π_c are such that $\pi \geq 1 - \pi_c = \delta$ and **π is comonotonic with δ** (they induce the same order on the referential according to $\leq_R$).
- $\mathcal{P} = \mathcal{P}(\pi) \cap \mathcal{P}(\pi_c)$
- **Theorem**: a generalized p-box is a belief function (random set) with focal sets

  $\{x: \pi(x) \geq \alpha\} \setminus \{x: \delta(x) > \alpha\}$

From generalized p-boxes to clouds

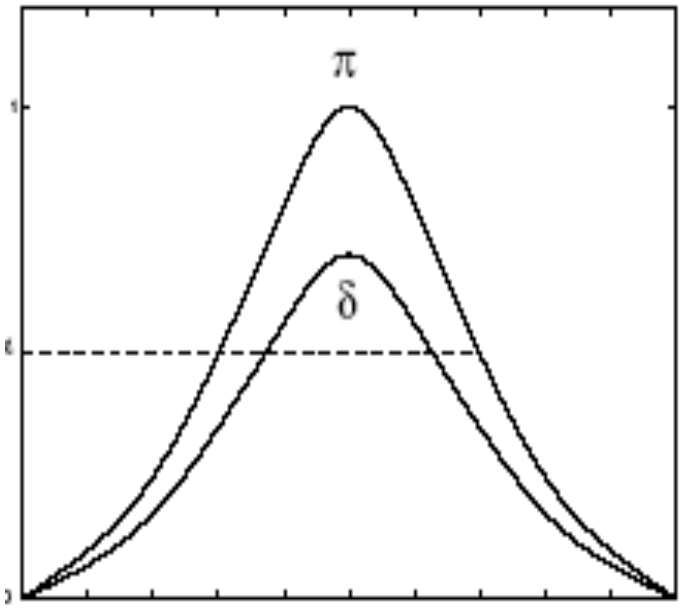


Fig 1.A Comonotonic cloud

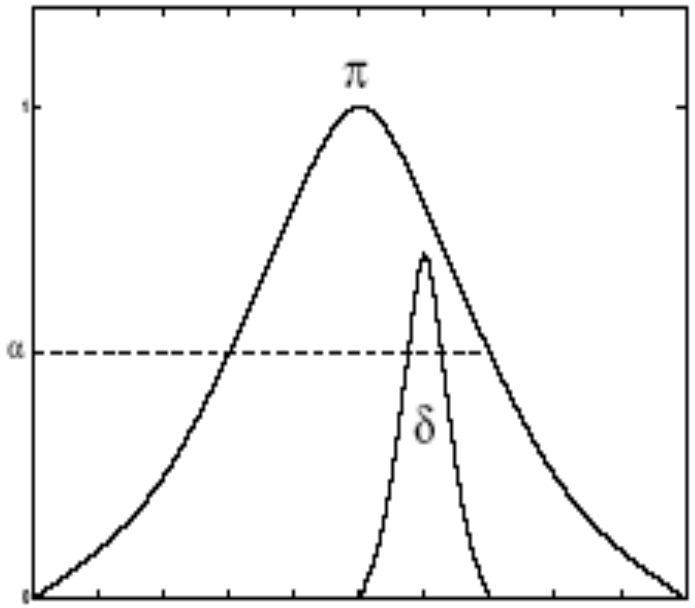


Fig 1.B Non-comonotonic cloud

CLOUDS

- Neumaier (2004) proposed a generalized interval as a pair of distributions ($\pi \geq \delta$) on a referential representing the family of probabilities $\mathcal{P} =$
  $\{P, \text{s. t. } P(\{x: \delta(x) > \alpha\}) \leq \alpha \leq P(\{x: \pi(x) \geq \alpha\}) \; \forall \alpha > 0\}$
- Distributions π and $1-\delta$ are possibility distributions such that $\mathcal{P} = \mathcal{P}(\pi) \cap \mathcal{P}(1-\delta)$
- It does not correspond to a belief function, not even a convex (2-monotone) capacity

SPECIAL CLOUDS

- Clouds are modelled by interval-valued fuzzy sets
- Comonotonic clouds = generalized p-boxes
- Fuzzy clouds: $\delta = 0$; they are possibility distributions

- Thin clouds: $\pi = \delta$:
  - Finite case : empty
  - Continuous case : there is an infinity of probability distributions in $\mathcal{P}(\pi) \cap \mathcal{P}(1-\pi)$ for bell-shaped π
  - Increasing π: only one probability measure p (π = cumulative distribution of p)

Probability intervals

- Probability intervals = a finite collection L of imprecise assignments $[l_i , u_i]$ attached to elements s_i of a finite set S.
- A collection $L = \{[l_i , u_i]\ i = 1,\ldots n\}$ induces the family $\mathcal{P}_L = \{P: l_i \leq P(\{s_i\}) \leq u_i\}$.
- Lower/upper probabilities on events are given by
  - $P_*(A) = \max(\Sigma_{si\in A}\ l_i\ ;\ 1 - \Sigma_{si\notin A}\ u_i)$;
  - $P^*(A) = \min(\Sigma_{si\in A}\ u_i\ ;\ 1 - \Sigma_{si\notin A}\ l_i)$
- P_* is a 2-monotone Choquet capacity

How useful are these representations:

- P-boxes can address questions about threshold violations ($x \geq a$??), not questions of the form $a \leq x \leq b$
- The latter questions are better addressed by possibility distributions or generalized p-boxes

Relationships between representations

- Generalized p-boxes are special random sets that generalize BOTH p-boxes and possibility distributions
- Clouds extend GP-boxes but induce lower probabilities that are not even 2-monotonic.
- Probability intervals are not comparable to generalized p-boxes: they induce lower probabilities that are 2-monotonic

Important pending theoretical issues

- Comparing representations in terms of **informativeness**.
- **Conditioning** : several definitions for several purposes.
- **Independence notions**: distinguish between epistemic and objective notions.
- Find a general setting for **information fusion** operations (e.g. Dempster rule of combination).

Comparing belief functions in terms of informativeness

- **Consonant case** : relative specificity.

π' more specific (more informative) than π in the wide sense if and only if $\pi' \leq \pi$.

(any possible value in information state π' is at least as possible in information state π)

- Complete knowledge: $\pi(s_0) = 1$ and $= 0$ otherwise.
- Ignorance: $\pi(s) = 1, \forall\ s \in S$

Comparing belief functions in terms of informativeness

- Using contour functions:

$$\pi(s) = Pl(s) = \sum_{x \in E} m(E)$$

m_1 is more cf-informative that m_2 iff $\pi_1 \leq \pi_2$

- Using belief or plausibility functions :

m_1 is more pl-informative that m_2 iff $Pl_1 \leq Pl_2$

iff $Bel_1 \geq Bel_2$

It corresponds to comparing credal sets $\mathsf{P}(m)$:

$Pl_1 \leq Pl_2$ if and only if $\mathsf{P}(m_1) \subseteq \mathsf{P}(m_2)$

Specialisation

- m_1 is more specialised than m_2 if and only if
  - Any focal set of $m_1$is included in at least one focal set of m_2
  - Any focal set of m_2 contains at least one focal set of m_1
  - There is a stochastic matrix W that shares masses of focal sets of m_2 among focal sets of m_1 that contain them:
- $$m_2(E) = \sum_{F \subseteq E} w(E, F)\, m_1(F)$$

Results

- $m_1 \subseteq_s m_2$ implies $m_1 \subseteq_{Pl} m_2$ implies $m_1 \subseteq_{cf} m_2$
- Typical information ordering for belief functions : $m_1 \subseteq_s m_2$ iff $Q_1 \leq Q_2$
- $m_1 \subseteq_s m_2$ implies $m_1 \subseteq_Q m_2$ implies $m_1 \subseteq_{cf} m_2$
- However $m_1 \subseteq_{Pl} m_2$ and $m_1 \subseteq_Q m_2$ are not comparable and can contradict each other
- In the consonant case : all orderings collapse to $m_1 \subseteq_{cf} m_2$

Example

- $S = \{a, b, c\}$; $m_1(ab) = 0.5$, $m_1(bc) = 0.5$;
- $m_2(abc) = 0.5$, $m_2(b) = 0.5$
- $m_1 \subseteq_s m_2$ nor $m_2 \subseteq_s m_1$ hold
- $m_2 \subset_{Pl} m_1 : Pl_1(A) = Pl_2(A)$
  but $Pl_2(ac) = 0.5 < Pl_1(ac) = 1$
- $m_1 \subset_Q m_2 : Q_1(A) = Q_2(A)$
  but $Q_1(ac) = 0 < Q_2(ac) = 0.5$
- And contour functions are equal : a/0.5, b/1, c/0.5

Conditional probability

- Querying a generic probability based on sure singular information:
  - P represents generic information (statistics over a population),
  - C represents singular evidence (variable instantiation for a case x at hand)
  - The relative frequency P(B|C) is used as the degree of belief that $x \in C$ satisfies B.

Conditional probability

- Revising a subjective probability
  - P(A) represents singular information, an agent's prior belief on what is the current state of the world (that a birth date $x \in A$…).
  - C represents an additional sure information about the value of x : $x \in C$ for sure.
  - P(A|C) represents the agent's posterior belief that $x \in A$.

Conditioning a credal set

- *Let $\mathcal{P}$ be a credal set representing generic information and C an event*
- *Two types of processing :*

***1. Querying** : C represents available singular facts:*
compute the degree of belief in A in context C as
$Cr(A \mid C) = Inf\{P(A \mid C), P \in \mathcal{P}, P(C) > 0\}$ (Walley).

***2. Revision** : C represents a set of universal truths;*
Add P(C) = 1 to the set of conditionals $\mathcal{P}$.
Now we must compute $Cr(A||C) = Inf\{P(A)\ P \in \mathcal{P}, P(C) = 1\}$
If P(C) = 1 is incompatible with $\mathcal{P}$, use maximum likelihood:
$Cr(A||C) = Inf\{P(A|C)\ P \in \mathcal{P}, P(C)$ maximal $\}$

Example : $A \rightleftarrows B \longrightarrow C$

- $\mathcal{P}$ *is the set of probabilities such that*
  - $P(B|A) \geq \alpha$ *Most A are B*
  - $P(C|B) \geq \beta$ *Most B are C*
  - $P(A|B) \geq \gamma$ *Most B are A*
- ***Querying on context A :*** *Find the most narrow interval for P(C|A) (Linear programming): we find*

$$P(C|A) \geq \alpha \cdot max(0,\ 1 - (1 - \beta)/\gamma)$$

  - *Note : if* $\gamma = 0$ *, P(C|A) is unknown even if* $\alpha = 1$*.*
- ***Revision:*** *Suppose* $P(A) = 1$*, then* $P(C||A) \geq \alpha \cdot \beta$
  - *Note:* $\beta > max(0,\ 1 - (1 - \beta)/\gamma)$
- **Revision improves generic knowledge, querying does not.**

CONDITIONING RANDOM SETS AS IMPRECISE PROBABILISTIC INFORMATION

- A disjunctive random set $(\mathcal{F}, m)$ representing background knowledge is equivalent to a set of probabilities $\mathcal{P} = \{P: \forall A, P(A) \geq Bel(A)\}$ (*NOT conversely*).
- Querying this information based on evidence C comes down to performing a sensitivity analysis on the conditional probability $P(\cdot|C)$
  - $Bel_C(A) = \inf \{P(A|C): P \in \mathcal{P}, P(C) > 0\}$
  - $Pl_C(A) = \sup \{P(A|C): P \in \mathcal{P}, P(C) > 0\}$

- **Theorem:** functions $Bel_C(A)$ and $Pl_C(A)$ are belief and plausibility functions of the form

  $$Bel_C(A) = Bel(C \cap A)/(Bel(C \cap A) + Pl(C \cap A^c))$$

  $$Pl_C(A) = Pl(C \cap A)/(Pl(C \cap A) + Bel(C \cap A^c))$$

  where $Bel_C(A) = 1 - Pl_C(A^c)$
- *This conditioning does not add information*:
- If $E \cap C \neq \emptyset$ and $E \cap C^c \neq \emptyset$, it is not clear how much mass must be transferred to $E \cap C$.
- If so for all $E \in \mathcal{F}$, then $m_C(C) = 1$ (the resulting mass function m_C expresses total ignorance on C)
  - **Example: If opinion poll yields:**
  - $m(\{a, b\}) = \alpha$, $m(\{c, d\}) = 1 - \alpha$,

  The proportion of voters for a candidate in $C = \{b, c\}$ is unknown.
  - *However if we hear a and d resign ($Pl(\{a, d\} = 0)$ then $m(\{b\}) = \alpha$, $m(\{c\}) = 1 - \alpha$* (Dempster conditioning, see further on)

CONDITIONING UNCERTAIN SINGULAR EVIDENCE

- A mass function *m on S, represents uncertain evidence*
- A new **sure** piece of evidence is viewed as a conditioning event C

1. *Mass transfer* : for all $E \in \mathcal{F}$, m(E) moves to $C \cap E \subseteq C$
   - The mass function after the transfer is $m_t(B) = \Sigma_{E : C \cap E = B}\, m(E)$
   - But the mass transferred to the empty set may not be zero!
   - $m_t(\varnothing) = Bel(C^c) = \Sigma_{E : C \cap E = \varnothing}\, m(E)$ is the degree of conflict with evidence C

3. *Normalisation*: $m_t(B)$ should be divided by

   $$Pl(C) = 1 - Bel(C^c) = \Sigma_{E : C \cap E \neq \varnothing}\, m(E)$$

- ***This is revision of an unreliable testimony by a sure fact***

DEMPSTER RULE OF CONDITIONING = PRIORITIZED MERGING

The conditional plausibility function Pl(.|C) is

$$Pl(A||C) = \frac{Pl(A \cap C)}{Pl(C)} \quad ; \quad Bel(A||C) = 1 - Pl(A^c||C)$$

- C surely contains the value of the unknown quantity described by m. **So $Pl(C^c) = 0$**
  - *The new information is interpreted as asserting the impossibility of C^c: Since C^c is impossible you can change $x \in E$ into $x \in E \cap C$ and transfer the mass of focal set E to $E \cap C$.*
- *The new information improves the precision of the evidence :* **This conditioning is different from Bayesian (Walley) conditioning**

EXAMPLE OF REVISION OF EVIDENCE : The criminal case

- **Evidence 1** : three suspects : Peter Paul Mary
- **Evidence 2** : The killer was randomly selected man vs.woman by coin tossing.
  - So, S = { Peter, Paul, Mary}
- TBM modeling: The masses are m({Peter, Paul}) = 1/2 ; m({Mary}) = 1/2
  - Bel(Paul) = Bel(Peter) = 0. Pl(Paul) = Pl(Peter) = 1/2
  - Bel(Mary) = Pl(Mary) = 1/2
- **Bayesian Modeling:** A prior probability
  - P(Paul) = P(Peter) = 1/4; P(Mary) = 1/2

- **Evidence 3** : Peter was seen elsewhere at the time of the killing.
- **TBM**: So Pl(Peter) = 0.
  - m({Peter, Paul}) = 1/2; m_t({Paul}) = 1/2
  - *A uniform probability on {Paul, Mary} results.*
- **Bayesian Modeling:**
  - P(Paul | not Peter) = 1/3; P(Mary | not Peter) = 2/3.
  - A very debatable result that depends on where the story starts. *Starting with i males and j females:*
    - P(Paul | Paul OR Mary) = j/(i + j);
    - P(Mary | Paul OR Mary) = i/(i + j)
- *Walley conditioning:*
  - Bel(Paul) = 0; Pl(Paul) = 1/2
  - Bel(Mary) = 1/2; Pl(Mary) = 1

Information fusion

- Dempster rule of combination in evidence theory:
  - independent sources, normalised or not
  - Does nor preserve consonance of inputs
  - No well-accepted idempotent fusion rule.
- In possibility theory : many fusion rules.
  - The minimum rule : idempotent (= minimal commitment fusion rule for consonant belief functions, not for other ones)
  - The product rule : coincides with the contour function obtained from unnormalized Dempster rule applied to consonant belief functions

Information fusion

- Fusion with credal sets
  - The intersection of credal sets is a credal set.
  - The lower probability bounds obtained by the intersection of credal sets induced from belief functions : $P^*(A) = \inf\{P(A), P \geq Bel_1, P \geq Bel_2\}$ is not a belief function.
  - The set function $g(A) = \max(Bel_1(A), Bel_2(A))$ is not a belief function

Decision with imprecise probability techniques

- Accept incomparability when comparing imprecise utility evaluations of decisions.
  - Pareto optimality : decisions that dominate other choices for all probability functions
  - E-admissibility : decisions that dominate other choices for at least one probability function (Walley, etc…)
- Select a single utility value that achieves a compromise between pessimistic and optimistic attitudes.
  - Select a single probability measure (Shapley value = pignistic transformation) and use expected utility (SMETS)
  - Compare lower expectations of decisions (Gilboa)
  - Generalize Hurwicz criterion to focal sets with degree of optimism (Jaffray)

Conclusion

- *There exist a coherent range of uncertainty theories combining interval and probability representations.*
  - Imprecise probability is the proper theoretical umbrella
  - The choice between subtheories depends on how expressive it is necessary to be in a given application.
  - There exists simple practical representations of imprecise probability : possibility theory is the simplest, and belief functions are a good compromise between calculability and expressivity
- *Discrepancies between the theories remain on conditioning, combination rules, because their language primitives differ:*
  - One cannot obviously express concepts defined by probability masses using credal sets, and conversely, let alone possibility distributions.